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Realizations of Causal Manifolds by Quantum Fields

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Quantum mechanical operators and quantum fields are interpreted as realizations of timespace manifolds. Such causal manifolds are parametrized by the classes of the positive unitary operations in all complex operations, i.e., by the homogenous spaces $\mathbf{D}(n) = \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^{n})/\mathbf{U}(n)$ with $n = 1$ for mechanics and $n = 2$ for relativistic fields. The rank n gives the number of both the discrete and continuous invariants used in the harmonic analysis, i.e., two characteristic masses in the relativistic case. 'Canonical' field theories with the familiar divergencies are inappropriate realizations of the real 4-dimensional causal manifold $\mathbf{D}(2)$. Faithful timespace realizations do not lead to divergencies. In general they are reducible, but nondecomposable—in addition to representations with eigenvectors (states, particle), they incorporate principal vectors without a particle (eigenvector) basis as exemplified by the Coulomb field.

INTRODUCTION

Quantum theory deals primarily with operations, e.g., timespace translations and rotations or hypercharge and isospin transformations. Its experimental interpretation relies on states or particles, i.e., eigenvectors of asymptotically relevant operations. The particle properties (mass, spin, charge etc.) we are measuring are the corresponding eigenvalues. The modality structure of quantum theory, e.g., the probability amplitudes, is formulated using complex linear structures with conjugations. Therefore, complex linear operations and their unitary substructures—not necessarily positive unitary[2]—play the central role in quantum theory.

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[2] In the *orthogonal* and *unitary* groups $\mathbf{O}(N_+, N_-)$ and $\mathbf{U}(N_+, N_-)$, respectively, the *positive* orthogonal and unitary ones are $\mathbf{O}(N)$ and $\mathbf{U}(N)$, respectively.

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The characterization of particles as positive unitary irreducible representation of the Poincaré group by Wigner (1939) is an asymptotic experimentally oriented language. A particle language, i.e., an eigenvector basis, is too narrow to describe interactions. Following up the identity of particles in their interaction is impossible—one tries to evade or circumvent the particle-interaction complementarity using words and concepts like 'off shell' or 'virtual particles' or 'ghosts,' etc. There are interaction structures which do not show up in the projection to a particle basis.

First of all, the nontranslative homogeneous interaction symmetries are truly larger than particle symmetries. Concerning external operations, particles have homogeneous symmetry properties only with respect to the 'little groups' spin **SU**(2) or helicity (polarization) **U**(1), which are true subgroups of the interaction-compatible Lorentz $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)$ group. With respect to the internal operations as seen in the interactions of the 'standard model of elementary particles,' there remains for particles only an abelian **U**(1) electromagnetic symmetry, defined as the internal 'little group' (fixgroup) of the degenerated ground state, as subgroup of the hyperisospin transformation group **U**(2). If color **SU**(3) confinement holds, only color singlets arise as particles.

In addition to these projections from the 'large' interaction symmetries to the 'little' particle symmetries, both for external and internal operations, there are operational structures in the dynamics without any asymptotic particle trace, the most prominent ones given by the Coulomb and gauge degrees of freedom of the standard interactions, formalized in relativistic nonabelian quantum field theory in cooperation with Fadeev–Popov degrees of freedom (Becchi *et al.*, 1976; Nakanishi and Ojima, 1990). All those 'ghost' structures come in connection with *indefinite unitary* representations of timespace translations (Saller, n.d.) ultimately tied to the indefinite structure of the noncompact Lorentz group **O**(1,3) which is indispensible for a nontrivial relativistic causal order.

This paper is an attempt to connect the asymptotic concepts 'particle' and even 'time' and 'space' with the interactions on a deeper operation-oriented (Finkelstein, 1996) level. Timespace will be formalized by a coset structure—as the noncompact real homogeneous space $\mathbf{D}(n) = \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)/\mathbf{U}(n)$ which is the manifold of the positive, unitary, probability amplitude-related operations $\mathbf{U}(n)$ in all complex linear ones. The causal manifold $\mathbf{D}(n)$ has real dimension n^2 and real rank n. The abelian case $n = 1$ involves the real 1-dimensional causal group (time) **D**(1) as the framework which is extensively used in quantum mechanics. The next simple case with timespace rank $n = 2$, the first nonabelian one, involves the real 4-dimensional homogeneous manifold **D**(2) with internal stability group **U**(2), it shows all the features familiar from relativistic quantum field theories.

The rank n of the causal manifold $\mathbf{D}(n)$ shows up in the number of real, continuous invariants used in its representations. The representations of the causal group $\mathbf{D}(1)$ are characterized by one continuous invariant (frequency) ω, which serves, e.g., as unit in energy eigenstates of a quantum mechanical dynamics, whereas the representations of $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$ for the homogeneous timespace $\mathbf{D}(2)$ involve two continuous invariants which may be tentatively called a particle mass m_1 and an interaction mass m_2—or m_1 and a dimensionless ratio m_2^2/m_1^2, possibly related to the coupling constants used for relativistic interactions.

A parametrization of timespace manifold $\mathbf{D}(n)$ realizations by vectors of complex linear representation spaces leads to the concepts of quantum mechanical operators for $n = 1$ and relativistic quantum fields for $n = 2$. In the latter case, the representation of the rank 2 operations in $\mathbf{D}(2)$—not only rank 1 (time translations) as done in conventional particle-oriented linear quantum fields—gives rise to a framework where product representations are definable without the light cone-supported divergencies found for interacting linear quantum fields.

1. "CANONICAL" QUANTIZATION

Quantum mechanics as a theory for time-dependent operators was very successfull. The quantization involved, called "canonical", was simply taken over—in a Lorentz-compatible extension—for timespace-dependent operators (quantum fields).

1.1. Quantum Structure in Mechanics

Heisenberg's noncommutativity condition $[iP, X] = 1$ (in units with $\hbar = 1$) for position–momentum operator pairs (X, P) is the trivial time $t = 0$ element of time representations[3] $[iP, X](t)$ by a quantum mechanical dynamics, e.g., in the—not only historically relevant—simplest example of the harmonic oscillator with Hamiltonian $H = P^2/(2M) + \kappa X^2/2$, involving a mass M and a spring constant κ or a frequency $m^2 = \kappa/M$ and an intrinsic length $l^4 = 1/\kappa M$:

$$\left.\begin{aligned} \frac{d}{dt}X(t) &= ml^2P(t) \\ \frac{d}{dt}P(t) &= -\frac{m}{l^2}X(t) \end{aligned}\right\} \Rightarrow [iP, X](t) = \cos mt \tag{1}$$

[3] The linear dependence is used in the notation $[a(y), b(x)] = [a, b](x - y)$, etc.

The eigenvectors of the time translations can be built by products of the creation–annihilation pair (u, u*)

$$\left.\begin{aligned} u &= \frac{1}{l\sqrt{2}} X - \frac{l}{\sqrt{2}} iP \\ u^\star &= \frac{1}{l\sqrt{2}} X + \frac{l}{\sqrt{2}} iP \end{aligned}\right\} \Rightarrow \begin{aligned} &\frac{d}{dt}(u^\star)^k(u)^l(t) = i(l-k)m(u^\star)^k(u)^l(t) \\ &\text{with } k, l \in \mathbb{N} \end{aligned} \tag{2}$$

The harmonic oscillator as **U**(1)-time representation uses a point measure for the frequencies as the linear forms on the time translations:

$$\begin{aligned} \mathbf{U}(1) \ni [u^\star, u](t) &= e^{imt} \\ &= \int d\mu\, \delta(\mu - m) e^{i\mu t} \\ &= \frac{\epsilon(t)}{i\pi} \int d\mu\, \frac{\mu + m}{\mu^2 - m_P^2} e^{i\mu t} = \frac{1}{2i\pi} \oint \frac{d\mu}{\mu - m} e^{i\mu t} \end{aligned} \tag{3}$$

Here m_p denotes the principal value integration and $\oint$ the mathematically positive circle around all poles in the complex frequency plane $\mu \in \mathbb{R} \subset \mathbb{C}$.

The self-dual time representation[4] in **SO**(2) by position and momentum reads

$$\mathbf{SO}(2) \ni \begin{pmatrix} [iP, X] & [X, X] \\ [P, P] & [X, -iP] \end{pmatrix}(t) = \begin{pmatrix} \cos mt & il^2 \sin mt \\ \frac{i}{l^2} \sin mt & \cos mt \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \cos mt \\ i \sin mt \end{pmatrix} &= \int d\mu\, \delta(\mu^2 - m^2)\epsilon(\mu) \begin{pmatrix} \mu \\ m \end{pmatrix} e^{i\mu t} \\ &= \int d\mu\, \delta(\mu^2 - m^2)\epsilon(m) \begin{pmatrix} m \\ \mu \end{pmatrix} e^{i\mu t} \\ &= \frac{\epsilon(t)}{i\pi} \int \frac{d\mu}{\mu^2 - m_P^2} \begin{pmatrix} \mu \\ m \end{pmatrix} e^{i\mu t} = \frac{1}{i\pi} \oint \frac{d\mu}{\mu^2 - m^2} \begin{pmatrix} \mu \\ m \end{pmatrix} e^{i\mu t} \end{aligned} \tag{4}$$

In general, a dynamics is not linear, i.e., the Hamiltonian is not quadratic in the fundamental operator pairs (X, P), e.g., for the hydrogen atom with

[4] The **SO**(2)-matrices

$$\begin{pmatrix} \cos mt & e^{\alpha} \sin mt \\ -e^{-\alpha} \sin mt & \cos mt \end{pmatrix}$$

with $\alpha \in \mathbb{C}$ are equivalent to the familiar **SO**(2)-matrices with $\alpha = 0$.

the Hamiltonian $H = \vec{P}^2/(2M) - g_0/|\vec{X}|$. In those 'truly interacting' cases, the operators for the energy eigenstates may be complicated combinations of positions X and momenta P.

1.2. Distributive Quantization of Particle Fields

The relativistic embedding of the **SO(2)** time representations leads to two different results: Since the energy is embedded in a Lorentz energy–momentum vector $\mu \hookrightarrow (q^k)_{k=0}^3$, one obtains both 'scalar and vector cosinus and sinus'

$$\frac{d}{dt}\begin{pmatrix} \cos mt \\ i \sin mt \end{pmatrix} = im\begin{pmatrix} i \sin mt \\ \cos mt \end{pmatrix} \hookrightarrow \begin{cases} \partial_k\begin{pmatrix} \mathbf{c}^k(m|x) \\ i\mathbf{s}(m|x) \end{pmatrix} = im\begin{pmatrix} i\mathbf{s}(m|x) \\ c_k(m|x) \end{pmatrix} \\ \partial_k\begin{pmatrix} \mathbf{C}(m|x) \\ i\mathbf{S}^k(m|x) \end{pmatrix} = im\begin{pmatrix} i\mathbf{S}_k(m|x) \\ \mathbf{C}(m|x) \end{pmatrix} \end{cases} \tag{5}$$

which both fulfill a homogeneous Klein–Gordon equation

$$\left(\frac{d^2}{dt^2} + m^2\right)\begin{pmatrix} \cos mt \\ i \sin mt \end{pmatrix} = 0 \hookrightarrow \begin{cases} (\partial^2 + m^2)\begin{pmatrix} \mathbf{c}^k(m|x) \\ i\mathbf{s}(m|x) \end{pmatrix} = 0 \\ (\partial^2 + m^2)\begin{pmatrix} \mathbf{C}(m|x) \\ i\mathbf{S}^k(m|x) \end{pmatrix} = 0 \end{cases} \tag{6}$$

The embedding with an ordered Dirac energy–momentum measure at $q^2 = m^2$ for the translation eigenvectors e^{iqx}

$$\begin{aligned} \begin{pmatrix} \mathbf{c}^k(m|x) \\ i\mathbf{s}(m|x) \end{pmatrix} &= \int \frac{d^4q}{(2\pi)^3}\, \delta(q^2 - m^2)\epsilon(q_0)\begin{pmatrix} q^k \\ m \end{pmatrix} e^{ixq} \\ &= \frac{\epsilon(x_0)}{i\pi} \int \frac{d^4q}{(2\pi)^3}\, \frac{1}{q^2 - m_P^2}\begin{pmatrix} q^k \\ m \end{pmatrix} e^{ixq} \end{aligned} \tag{7}$$

defines the causally supported quantization distributions

$$\begin{pmatrix} \mathbf{c}^k(m|x) \\ i\mathbf{s}(m|x) \end{pmatrix} = 0 \qquad \text{for spacelike} \quad x^2 < 0 \tag{8}$$

The embedding with a 'not ordered' measure

$$\begin{pmatrix} \mathbf{C}(m|x) \\ i\mathbf{S}^k(m|x) \end{pmatrix} = \int \frac{d^4q}{(2\pi)^3}\, \delta(q^2 - m^2)\epsilon(m)\begin{pmatrix} m \\ q^k \end{pmatrix} e^{ixq} \tag{9}$$

defines the Fock state functions, which have also nontrivial spacelike contributions.

To illustrate quantum fields with linear equations of motion and particle interpretation, following the harmonic oscillator linear quantum structures, a free Dirac field $\boldsymbol{\Psi}$ for particles with mass $m \neq 0$, e.g., the electron, yields a good example:

$$\left(i\frac{d}{dt} + m\right)\mathrm{u}(t) = 0 \hookrightarrow (i\gamma_k\partial^k + m)\boldsymbol{\Psi}(x) = 0 \tag{10}$$

Weyl fields arise for $m = 0$ with a left- or right-handed Weyl representation $\frac{1}{2}(1 + \gamma_5)\gamma_k \cong \rho_k = (\mathbf{1}, \vec{\sigma})$ and $\frac{1}{2}(1 - \gamma_5)\gamma_k \cong \check{\rho}_k = (\mathbf{1}, -\vec{\sigma})$, resp., replacing the Dirac representation.

The Feynman propagator is the Fock value $\langle\ldots\rangle$ of the sum of the causally ordered quantization anticommutator and the commutator

$$\begin{aligned}\langle\mathscr{C}\overline{\boldsymbol{\Psi}}\boldsymbol{\Psi}\rangle(x) &= \frac{i}{\pi}\int\frac{d^4q}{(2\pi)^3}\frac{\gamma_k q^k + m}{q^2 - m^2 + io}e^{iqx}\\ &= \langle -\epsilon(x_0)\{\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}\}(x) + [\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}](x)\rangle\end{aligned} \tag{11}$$

The Fock form function with its spacelike contributions

$$\begin{aligned}\langle[\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}]\rangle(x) &= \mathbf{Exp}(im|x) = \mathbf{C}(m|x) + i\gamma_k\mathbf{S}^k(m|x)\\ &= \int\frac{d^4q}{(2\pi)^3}\epsilon(m)(\gamma_k q^k + m)\delta(q^2 - m^2)e^{ixq}\end{aligned} \tag{12}$$

will be discussed later (Section 5.3). The distribution for the quantization anticommutator

$$\begin{aligned}\{\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}\}(x) &= \mathbf{exp}(im|x) = \gamma_k\mathbf{c}^k(m|x) + i\mathbf{s}(m|x)\\ &= \int\frac{d^4q}{(2\pi)^3}\epsilon(q_0)(\gamma_k q^k + m)\delta(q^2 - m^2)e^{ixq}\\ &= \frac{\epsilon(x_0)}{i\pi}\int\frac{d^4q}{(2\pi)^3}\frac{\gamma_k q^k + m}{q^2 - m_P^2}e^{ixq}\end{aligned} \tag{13}$$

is the interaction relevant structure, which causes the 'divergencies' in relativistic field theories (not the particle relevant Fock value of the commutator). The distinction between $\mathbf{exp}(im|x)$ and $\mathbf{EXP}\ (im|x)$ is characteristic for the relativistic case with its causal partial order.

The quantization distribution is given explicitly in timespace coordinates

$$\begin{aligned}\mathbf{c}^k(m|x) &= \frac{m^2}{8\pi}\frac{m^2\epsilon(x_0)x^k}{2}\left[\delta'\left(\frac{m^2x^2}{4}\right) - \delta\left(\frac{m^2x^2}{4}\right) + \frac{1}{2}\vartheta(x^2)\mathscr{E}_2(m^2x^2)\right]\\ \mathbf{s}(m|x) &= \frac{m^2}{8\pi}m\epsilon(x_0)\left[\delta\left(\frac{m^2x^2}{4}\right) - \vartheta(x^2)\mathscr{E}_1(m^2x^2)\right]\end{aligned} \tag{14}$$

with $\mathbf{c}^k(0|x) = [\epsilon(x_0)x^k/\pi]\,\delta'(x^2)$. These distributions involve the functions $\mathscr{E}_n$, in general for $n = 0, 1, \ldots$, with $\tau^2 = \vartheta(x^2)m^2x^2$ ('eigentime' τ), Bessel coefficients J_n (Sneddon, 1961), and the beta-function B,

$$\mathscr{E}_n(\tau^2) = n!\,\frac{J_n(\tau)}{(\tau/2)^n} = n!\sum_{j=0}^{\infty}\frac{(-\tau^2/4)j}{j!(j+n)!} = \frac{\int_0^1 d\mu^2(\mu^2)^{-1/2}(1-\mu^2)^{-1/2+n}\cos\mu\tau}{B(\frac{1}{2},\frac{1}{2}+n)}$$

$$\mathscr{E}_{n+1}(\tau^2) = -(n+1)\frac{d\mathscr{E}_n(\tau^2)}{d\tau^2/4},\qquad \mathscr{E}_n(0) = 1 \tag{15}$$

The vectorial distribution for time $x_0 = 0$ describes the quantization of linear fields

$$\begin{aligned}\mathbf{exp}(im|x)|_{x_0=0} &= \gamma_k\mathbf{c}^k(m|\mathbf{x}) = \gamma_0\delta^3(\mathbf{x})\\ \int d^3x\,\{\overline{\mathbf{\Psi}}, \mathbf{\Psi}\}(x)|_{x_0=0} &= \gamma_0\end{aligned} \tag{16}$$

The fact that the linear quantization gives no functions, e.g., that $\mathbf{exp}(im|0)$ does not make sense or that two quantizations cannot be simply multiplied for a product representation of the translations, e.g., $\mathbf{exp}(im|x)\ \mathbf{exp}(im|x)$ arising in a perturbative approach (vacuum polarization in quantum electrodynamics) illustrates the familiar divergence problem for linear fields, which has to be treated by sophisticated techniques. It hints at the inappropriateness of particle fields for more than a perturbative formulation of interactions, e.g., their inappropriateness to parametrize a bound-state problem.

Using a decomposition of all translations into time and space translations, e.g., given in a massive particle rest system, the time-ordered integral of the spacelike trivial quantization distributions gives—after interchanging time and energy integration—the space-dependent interaction functions, e.g., the Yukawa interaction and force

$$\begin{aligned}\frac{1}{2}\int dx_0\,\epsilon(x_0)\mathbf{s}(m|x) &= md(m, |\vec{x}|)\\ &= m\int\frac{d^3q}{(2\pi)^3}\frac{e^{-i\vec{q}\vec{x}}}{\vec{q}^2+m^2} = \frac{m}{4\pi|\vec{x}|}e^{-m|\vec{x}|}\\ \frac{1}{2}\int dx_0\,\epsilon(x_0)\gamma_k\mathbf{c}^k(m|x) &= \gamma_a\partial^a d(m, |\vec{x}|)\\ &= -\vec{\gamma}\vec{x}\,\frac{1+m|\vec{x}|}{4\pi|\vec{x}|^3}e^{-m|\vec{x}|}\end{aligned} \tag{17}$$

The singularities at $\vec{x} = 0$ reflect the divergence problems of linear fields, if they are used to parametrize interactions.

Interchanging space and momenta integrations, the space integrals of the quantization distributions lead to the time development of the harmonic oscillator

$$\int d^3x \gamma_k \mathbf{c}^k(m|x) = \gamma_0 \cos mx_0, \qquad \int d^3x \, \mathbf{s}(m|x) = \sin mx_0 \tag{18}$$

The distributive quantization of relativistic fields identifies the eigenvalue of the harmonic oscillator time translation dependence, given by the frequency m_0, with the characteristic mass for the Yukawa interaction space dependence, given by the inverse range m_r

$$\left(\frac{d^2}{dt^2} + m_0^2\right)\cos m_0 t = 0, \qquad \left(-\frac{\partial^2}{\partial \vec{x}_2} + m_r^2\right)d(m_r, |\vec{x}|) = \delta(\vec{x}) \tag{19}$$

in the Lorentz-compatible timespace translation dependence

$$\text{with } m_0 = m_r\text{:} \quad (\partial^2 + m^2)\mathbf{c}^k(m|x) = 0 \tag{20}$$

2. CAUSAL TIMESPACES

Observables; in mechanics depend on time coordinates, relativistic fields depend on timespace coordinates. In this section, timespaces are reformulated in a general symmetry-oriented algebraic framework starting from complex linear operations.

2.1. Causal Complex Algebras

Cartan's representation of the timespace translations (real 4-dimensional Minkowski vector space) uses hermitian complex 2×2 matrices from the associative unital algebra of all complex 2×2 matrices. Together with the time translations (real numbers) of mechanics, they are given as follows:

$$x = \begin{cases} t = \bar{t} & \in \mathbf{C}(1) \\ \dfrac{1}{2}\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x^\star & \in \mathbf{C}(2) \end{cases} \tag{21}$$

Those familiar cases should be used as illustrations for the general case working with complex $n \times n$ matrices for $n \geq 1$.

The matrices $z \in \mathbf{C}(n)$ with the usual hermitian conjugation $*$ are an involutive algebra, decomposable into two isomorphic real vector spaces of dimension n^2.

$$\begin{aligned} \mathbb{C}(n) &= \mathbb{R}(n) \oplus i\mathbb{R}(n) \cong \mathbb{R}^{2n^2}, \qquad z = x + iy \\ \mathbb{R}(n) &= \{x \in \mathbb{C}(n) | x = x^\star\} \cong \mathbb{R}^{n^2} \end{aligned} \tag{22}$$

The symmetric vector subspace will be called the *matrix representation* $\mathbb{R}(n)$ *of the timespace translations* with n the *rank of timespace.*

A basis for $\mathbb{C}(n)$ is given by

$$[\rho(n)^j, i\rho(n)^j]_{j=0}^{n^2-1} \quad \text{with} \quad \{\rho(n)^j\}_{j=0}^{n^2-1} = \left\{\frac{\mathbf{1}(n)}{n}, \frac{\sigma(n)^a}{2}\right\}_{a=1}^{n^2-1} \tag{23}$$
$$z = z_j\rho(n)^j, \qquad z_A^{\dot{A}} = z_j[\rho(n)^j]_A^{\dot{A}}, \qquad A, \dot{A} = 1, \ldots, n$$

using the unit matrix $\mathbf{1}(n)$ and $(n^2 - 1)$ generalized hermitian traceless Pauli matrices $\sigma(n)^a$ for $n \geq 2$, i.e., three Pauli matrices $\vec{\sigma}$ for $n = 2$, eight Gell-Mann matrices $\sigma(3)^a = \lambda^a$ for $n = 3$, etc.:

$$\begin{aligned} &\sigma(n)^a = \sigma(n)^{a\star}, \qquad \operatorname{tr} \sigma(n)^a = 0 \\ &[i\sigma(n)^a, i\sigma(n)^b] = \alpha^{abc} i\sigma(n)^c, \qquad \text{totally antisymmetric } \alpha^{abc} \in \mathbb{R} \\ &\{\sigma(n)^a, \sigma(n)^b\} = 2\delta^{ab}\mathbf{1}(n) + \delta^{abc}\sigma(n)^c, \qquad \text{totally symmetric } \delta^{abc} \in \mathbb{R} \end{aligned} \tag{24}$$

The determinant defines the *abelian projection* of $\mathbb{C}(n)$ to the complex numbers $\mathbb{C}(1) = \mathbb{R} \oplus i\mathbb{R}$, compatible with the unital multiplication and the conjugation (a *-monoid morphism)

$$\det: \quad \mathbb{C}(n) \to \mathbb{C}, \quad z \mapsto z^n = \det z, \qquad \begin{cases} \det z \circ z' = \det z \det z' \\ \det \mathbf{1}(n) = 1 \\ \det z^\star = \overline{\det z} \end{cases} \tag{25}$$

By polarization, i.e., by combining appropriately $(z_1 \pm z_2 \pm \cdots \pm z_n)^n$, one obtains a totally symmetric multilinear form, familiar for the Minkowski translations $\mathbb{R}(2)$ as Lorentz bilinear form with indefinite signature

$$\begin{aligned} \eta: \quad &\mathbb{C}(n) \times \cdots \times \mathbb{C}(n) \to \mathbb{C} \\ &(z_1, \ldots, z_n) \mapsto \eta(z_1, \ldots, z_n) = \epsilon^{A_1 \cdots A_n} \epsilon_{\dot{A}_1 \cdots \dot{A}_n} (z_1)_{A_1}^{\dot{A}_1} \cdots (z_n)_{A_n}^{\dot{A}_n} \\ n = 1: \quad &\eta(z) = \det z = z \\ n = 2: \quad &\eta(z_1, z_2) = (z_1 + z_2)^2 - (z_1 - z_2)^2, \qquad \operatorname{sign} \eta|_{R(2)} = (1, 3) \end{aligned} \tag{26}$$

The trace part and the traceless part of a translation will be called a time translation in $\mathbb{T} \cong \mathbb{R}$ and a space translation in $\mathbb{S}(n) \cong \mathbb{R}^{n^2-1}$,

$$x = x_j\rho(n)^j = x_0 \frac{\mathbf{1}(n)}{n} + x_a \frac{\sigma(n)^a}{2} \tag{27}$$

However, a decomposition into time and space translations is incompatible with the determinant, since for $n \geq 2$ in general $\det(x + y) \neq \det x + \det y$.

The involutive algebra $\mathbb{C}(n)$ is a C^*-algebra (Bratelli and Robinson, 1979; Richart, 1960) with the norm $\|z\|^2 = \langle z|z\rangle$ induced by the scalar product

$$z, z' \in \mathbb{C}(n) \Rightarrow \langle z|z'\rangle = \operatorname{tr} z' \circ z^\star \tag{28}$$

Any C*-algebra is (partially) ordered via the spectrum

$$x \succcurlyeq 0 \Leftrightarrow x = x^\star \quad \text{and} \quad \operatorname{spec} x \subset \mathbb{R}^+ = \{\alpha \in \mathbb{R} | \alpha \geq 0\} \tag{29}$$

Therewith, $\mathbb{C}(n)$ will be called a *causal complex algebra.*

All timespace translations can be diagonalized with a real spectrum

$$\begin{aligned} &x \in \mathbb{R}(n) \quad \Rightarrow \quad \operatorname{spec} x = \{\xi | \det(x - \xi \mathbf{1}(n)) = 0\} \subset \mathbb{R} \\ &n = 1: \quad x \in \mathbb{R}(1), \qquad \operatorname{spec} x = \{\tau\} \\ &n = 2: \quad x \in \mathbb{R}(2), \qquad \operatorname{spec} x = \left\{\frac{x_0 \pm |\vec{x}|}{2}\right\} \end{aligned} \tag{30}$$

The n real spectral values $\{\xi_r\}_{r=1}^n$ of a timespace translation $x \in \mathbb{R}(n)$ will be called its *Cartan coordinates.*

A linear transformation $z \in \mathbb{C}(n)$ is diagonalizable, $z = u \circ \operatorname{diag}(z) \circ u^{-1}$ if, and only if, it is normal, $z \circ z^* = z^* \circ z$. The diagonalization transformation is unitary, $u^{-1} = u^*$. Therewith $x \in \mathbb{R}(n)$ and $l \in i\mathbb{R}(n)$ are diagonalizable, but not any $z \in \mathbb{C}(n)$.

The C*-algebra order generalizes the familiar order of the 1- and 4-dimensional translations. With one nontrivial positive causal vector c, positivity is expressable by positive c-projected products (n causal projections)

$$\begin{aligned} &c, x \in \quad \mathbb{R}(n), \qquad c \succ 0 \\ &x \succcurlyeq 0 \quad \Leftrightarrow x_c^r = \eta(\underbrace{x, \ldots, x}_{r}, \underbrace{c, \ldots, c}_{n-r}) \geq 0 \quad \text{for} \quad r = 1, \ldots, n \end{aligned} \tag{31}$$

The characteristic functions for the causal translations use the spectral values

$$\begin{aligned} &x \in \mathbb{R}(n), \qquad x^n = \det x = \prod_{r=1}^n \xi_r, \qquad \xi_r \in \operatorname{spec} x \\ &\vartheta(x) = \prod_{r=1}^n \vartheta(\xi_r) = \prod_{r=1}^n \vartheta(x_c^r), \qquad c \succ 0 \\ &\epsilon(x) = \vartheta(x) - \vartheta(-x) \end{aligned} \tag{32}$$

with the familiar example for Minkowski timespace with a time component $\eta(x, c) = x_0$

$$n = 2: \quad \begin{cases} \vartheta(x) = \vartheta(x_0 + |\vec{x}|)\vartheta(x_0 - |\vec{x}|) = \vartheta(x_0)\vartheta(x^2) \\ \epsilon(x) = \epsilon(x_0)\vartheta(x^2) \end{cases} \tag{33}$$

The vector space of all timespace translations $\mathbb{R}(n)$ is the union of the positive and the negative *causal cone* and the *spacelike* submanifold

$$\begin{aligned} \mathbb{R}(n) &= \mathbb{R}(n)_{\text{causal}} \cup \mathbb{R}(n)_{\text{space}}, \qquad \mathbb{R}(n)_{\text{causal}} \cap \mathbb{R}(n)_{\text{space}} = \{0\} \\ \mathbb{R}(n)_{\text{causal}} &= \mathbb{R}(n)^{+}_{\text{causal}} \cup \mathbb{R}(n)^{-}_{\text{causal}}, \qquad \mathbb{R}(n)^{+}_{\text{causal}} \cap \mathbb{R}(n)^{-}_{\text{causal}} = \{0\} \\ \mathbb{R}(n)^{+}_{\text{causal}} &= \{x \in \mathbb{R}(n) | \text{spec } x \subset \mathbb{R}^{+}\} = -\mathbb{R}(n)^{-}_{\text{causal}} \end{aligned} \tag{34}$$

All translations can be written as sum of a positive and a negative timelike translation

$$\mathbb{R}(n) = \{x_+ + x_- | x_+, -x_- \in \mathbb{R}(n)^{+}_{\text{time}}\} \tag{35}$$

The positive causal cone is the disjoint union

$$\mathbb{R}(n)^{+}_{\text{causal}} = \{0\} \uplus \mathbb{R}(n)^{+}_{\text{time}} \uplus \mathbb{R}(n)^{+}_{\text{light}} \tag{36}$$

of the trivial translation (tip of the cone), the strictly positive *timelike* translations (interior of the cone) where the spectrum does not contain 0

$$\mathbb{R}(n)^{+}_{\text{time}} = \{x \in \mathbb{R}(n)^{+}_{\text{causal}} | 0 \notin \text{spec } x\} \tag{37}$$

and the strictly positive *lightlike* translations (skin of the tipless cone) where 0 is a spectral value

$$\mathbb{R}(n)^{+}_{\text{light}} = \{x \in \mathbb{R}(n)^{+}_{\text{causal}} | x \neq 0, \qquad 0 \in \text{spec } x\} \tag{38}$$

For the 1-dimensional totally ordered translations $\mathbb{R}(1) = \mathbb{R}$, one has a trivial space $\mathbb{R}(1)_{\text{space}} = \{0\}$. The nontrivial spacelike manifold for $n \geq 2$ is the disjoint union of $(n - 1)$ submanifolds $\mathbb{R}(m, n - m)_{\text{space}}$ with m strictly positive and $n - m$ strictly negative Cartan coordinates

$$n \geq 2: \quad \mathbb{R}(n)_{\text{space}} \setminus \{0\} = \biguplus_{m=1}^{n-1} \mathbb{R}(m, n - m)_{\text{space}} \tag{39}$$

In the 1-dimensional case there is no light, $\mathbb{R}(1)^{\pm}_{\text{light}} = 0$. Light is a genuine nonabelian phenomenon, arising for $n \geq 2$. There, the strictly positive (negative) lightlike manifold is the disjoint union of $(n - 1)$ submanifolds $\mathbb{R}(m, n - 1 - m)^{\pm}_{\text{light}}$ with exactly m trivial and $n - 1 - m$ strictly positive (negative) Cartan coordinates

$$n \geq 2: \quad \mathbb{R}(n)^{\pm}_{\text{light}} = \biguplus_{m=1}^{n-1} \mathbb{R}(m, n - 1 - m)^{\pm}_{\text{light}} \tag{40}$$

The *proper time* (*'eigentime'*) *projection* is the causal projection of the translations to the real numbers $\mathbb{R}(1) = \mathbb{R}$,

$$\tau:\quad \mathbb{R}(n) \to \mathbb{R},\quad \tau(x) = \epsilon(x)\left|\det x^{\frac{1}{n}}\right| = \begin{cases} \tau & \text{for} \quad n = 1 \\ \epsilon(x_0)\vartheta(x^2)\sqrt{x^2} & \text{for} \quad n = 2 \end{cases} \tag{41}$$

In general for $n \geq 2$, it is not linear, $\tau(x + y) \neq \tau(x) + \tau(y)$.

2.2. Timespace Manifolds

The complex algebra $\mathbb{C}(n)$ with the commutator as Lie bracket is, on the one hand, as complex n^2-dimensional space, the rank n Lie algebra[5] of the complex Lie group $\mathbf{GL}(\mathbb{C}^n) \subset \mathbb{C}(n)$, and, on the other hand, as real $2n^2$-dimensional space, the rank $2n$ Lie algebra of the real group[6] $\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)$. The antisymmetric space $i\mathbb{R}(n)$ in $\mathbb{C}(n)$ is the rank n Lie algebra of the unitary group $\mathbf{U}(n)$:

$$\begin{aligned} \mathbb{C}(n) &= \log \mathbf{GL}(\mathbb{C}^n), & \mathbf{GL}(\mathbb{C}^n) &= \exp \mathbb{C}(n) \\ \mathbb{R}(n) \oplus i\,\mathbb{R}(n) &= \log \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n), & \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n) &= \exp[\mathbb{R}(n) \oplus i\mathbb{R}(n)] \\ i\mathbb{R}(n) &= \log \mathbf{U}(n), & \mathbf{U}(n) &= \exp i\mathbb{R}(n) \end{aligned} \tag{42}$$

The vector space of the timespace translations $\mathbb{R}(n)$ is isomorphic to the quotient of the full with respect to the unitary Lie algebra. Its exponent is isomorphic to the corresponding homogeneous space, the real n^2-dimensional manifold with the orbits $g\mathbf{U}(n)$ of the unitary group $\mathbf{U}(n)$ in the full group $\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)$

$$\mathbb{R}(n) \cong \log \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)/\log \mathbf{U}(n), \qquad \mathbf{D}(n) = \exp \mathbb{R}(n) \cong \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)/U(n) \tag{43}$$

The 'compact in complex manifold' $\mathbf{D}(n)$ will be called a *timespace manifold;* it is isomorphic to the strictly positive timelike translations

$$\mathbf{D}(n) = \{d = \mathrm{e}\,\mathrm{e}^* | \mathrm{e} \in \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)\} \cong \mathbb{R}(n)^+_{\text{time}} \tag{44}$$

i.e., the timespace manifold $\mathbf{D}(n)$ can be embedded into its tangent space $\mathbb{R}(n)$ where light and space translations arise as genuine tangent phenomena.

Timespace $\mathbf{D}(n)$ is totally semiordered $\sqsubseteq$ (transitive and reflexive) via the abelian projection onto the totally ordered group $\mathbf{D}(1)$,

$$\begin{aligned} &\det:\quad \mathbf{D}(n) \to \mathbf{D}(1),\quad d \mapsto \det d \\ &d_{1,2} \in D(n):\quad d_1 \sqsubseteq d_2 \Leftrightarrow \det d_1 \leq \det d_2 \end{aligned} \tag{45}$$

[5]The Lie algebra of the Lie group G is denoted by $\log G$.

[6]$\mathbb{C}_{\mathbb{R}}$ denotes the real 2-dimensional structure $\mathbb{R} \oplus i\mathbb{R}$ of the complex numbers $\mathbb{C}$.

Therewith, $\mathbf{D}(n)$ will be also called a *causal manifold* (Hilgert and Olafsson, 1997). The set of all semiorder induced equivalence classes is isomorphic to the causal group $\mathbf{D}(1)$ and carries a total order.

Also the name *modulus manifold* is justified for $\mathbf{D}(n)$ since it parametrizes the possible scalar products of the complex space $\mathbb{C}^n$ with basis $\{e^A\}_{A=1}^n$ where the C*-algebra $\mathbf{C}(n)$ is acting on

$$\mathbf{D}(n) \ni \mathrm{e}\,\mathrm{e}^* \cong \begin{pmatrix} \langle e^1|e^1\rangle & \cdots & \langle e^1|e^n\rangle \\ \cdots & \cdots & \cdots \\ \langle e^n|e^1\rangle & \cdots & \langle e^n|e^n\rangle \end{pmatrix} \tag{46}$$

The real Lie groups[7] for the timespace manifold involve as factors the *causal group* $\mathbf{D}(1)$ (direct factor, denoted by $\times$), the *phase group* $\mathbf{U}(1)$, and the *special groups* $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n)$ and $\mathbf{SU}(n)$, respectively. They will be called

$$\begin{array}{lrcl} \textit{external group:} & \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n) & = & \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) \circ \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n) \\ \textit{internal group:} & \mathbf{U}(n) & = & \mathbf{U}(1_n) \circ \mathbf{SU}(n) \\ \textit{timespace manifold:} & \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)/\mathbf{U}(n) & \cong & \mathbf{D}(1) \times \mathbf{SD}(n) \end{array} \tag{47}$$

The second direct factor $\mathbf{SD}(n) = \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n)/\mathbf{SU}(n)$ in the manifold will be called the *Sylvester or boost submanifold.* It is trivial only for the abelian case $n = 1$.

The groups $\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)$ (external) and $\mathbf{U}(n)$ (internal) have with the cyclotomic group $\mathbf{I}_n = \{\alpha \in \mathbb{C} | \alpha^n = 1\}$ the centers, phase correlation groups, and adjoint groups[8]

$$\begin{gathered} \text{centr } \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n) \cong \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) \qquad \text{centr } \mathbf{U}(n) \cong \mathbf{U}(1) \\ \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) \cap \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n) \cong \mathbf{I}_n, \qquad \mathbf{U}(1_n) \cap \mathbf{SU}(n) \cong \mathbf{I}_n \\ \text{Ad } \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n) \cong \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n)/\mathbf{I}_n, \qquad \text{Ad } \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n) \cong \mathbf{SU}(n)/\mathbf{I}_n \end{gathered} \tag{48}$$

As a homogeneous space, a timespace manifold has a nontrivial external action [from left on $g\mathbf{U}(n) \in \mathbf{D}(n)$] with the causal group and the adjoint external group

$$\text{external action on } \mathbf{D}(n) \text{ with } \mathbf{D}(1) \times \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^n)/\mathbf{I}_n \tag{49}$$

and a trivial action (from right) with the internal group (therefore the name 'internal').

[7] $\mathbf{U}(1_n) \cong \mathbf{U}(1)$ and $\mathbf{D}(1_n) \cong \mathbf{D}(1)$ denote the scalar phase and causal (dilatation) group in $\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^n)$. Here $\mathbf{1}(n)$ is the unit of the $\mathbb{C}^n$-automorphism groups.

[8] The adjoint group of a group is defined as the classes up to the center Ad $G = G/\text{centr } G$.

There is another, more familiar chain of causal timespace manifolds, characterized by the orthogonal real structures

$$s = 0: \quad \mathbf{D}(1), \qquad s \geq 1: \quad \mathbf{D}(1) \times \mathbf{SO}^+(1, s)/\mathbf{SO}(s) \tag{50}$$

with $s \geq 0$ space dimensions. This chain meets the $\mathbf{GL}(\mathbb{C}_\mathbb{R}^n)/\mathbf{U}(n)$ chain only for the two timespace dimensions $n^2 = 1 + s = 1, 4$. Obviously, the orthogonal structures involve an invariant bilinear form for all dimensions.

For $n = 2$ one has the isomorphy with the orthochronous Lorentz group $\mathbf{SO}^+(1,3)$ and the rotation group $\mathbf{SO}(3)$:

$$\begin{aligned} \mathbf{GL}(\mathbb{C}_\mathbb{R}^2)/\mathbf{GL}(\mathbb{C}_\mathbb{R}) &\cong \mathbf{SL}(\mathbb{C}_\mathbb{R}^2)/\mathbb{I}_2 \cong \mathbf{SO}^+(1, 3) \\ \mathbf{U}(2)/\mathbf{U}(1) &\cong \mathbf{SU}(2)/\mathbb{I}_2 \cong \mathbf{SO}(3) \\ \mathbf{D}(2) = \mathbf{GL}(\mathbb{C}_\mathbb{R}^2)/\mathbf{U}(2) &\cong \mathbf{D}(1) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}(3) \end{aligned} \tag{51}$$

If one visualizes the real 4-dimensional timespace manifold $\mathbf{D}(2)$ embedded as the strict future cone $\mathbb{R}(2)^+_{\text{time}}$ in the Minkowski translations $\mathbb{R}(2)$, this cone has to be foliated[9] with the hyperboloids $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$ (hyperbolic foliation). The causal group $\mathbf{D}(1)$ action on the manifold connects different hyperboloids by 'hyperbolic hopping,' whereas the orthochronous group $\mathbf{SO}^+(1, 1)$ action on the individual hyperboloids can be described as 'hyperbolic stretching'. The Minkowski translations $\mathbb{R}(2)$ as tangent structure of the timespace manifold $\mathbf{D}(2)$ can be visualized with a 3-dimensional tangent plane of a timelike hyperboloid $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$ and the tangent line of 'blowing up' or 'shrinking' this hyperboloid with $\mathbf{D}(1)$. The causal order is the 'foliation order' of the positive timelike hyperboloids.

2.3. The Rank of Timespace

A timespace translation is diagonalizable with a unitary matrix

$$x = x^* \in \mathbb{R}(n): \quad x = u(x) \circ \text{diag}(x) \circ u(x)^* \tag{52}$$

The relativistic case $n = 2$ uses, in addition to two Cartan coordinates, two polar coordinates (φ, θ) from the unit sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$

[9] Take the 3-dimensional projection with hyperboloids $\mathbf{SO}^+(1, 2)/\mathbf{SO}(2)$ with the 2-dimensional tangent planes.

$$n = 2: \quad \frac{1}{2}\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = u(x)\begin{pmatrix} \frac{x_0 + |\vec{x}|}{2} & 0 \\ 0 & \frac{x_0 - |\vec{x}|}{2} \end{pmatrix} u(x)^* \tag{53}$$

$$u(x) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{for} \quad \vec{x} = |\vec{x}|\begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

Timespace translations and timespace manifold are isomorphic as real manifolds

$$\begin{aligned} \exp: &\quad \mathbb{R}(n) \to \mathbf{D}(n), \quad x \mapsto e^x \\ \log: &\quad \mathbf{D}(n) \to \mathbb{R}(n), \quad d \mapsto \log d \end{aligned} \tag{54}$$

using the exponentation and logarithm of the diagonal matrices

$$\begin{aligned} x = u(x) \circ \operatorname{diag}(x) \circ u(x)^* &\Rightarrow e^x = u(x) \circ e^{\operatorname{diag}(x)} \circ u(x)^* \\ d = u(d) \circ \operatorname{diag}(d) \circ u(d)^* &\Rightarrow \log d = u(d) \circ \log \operatorname{diag}(d) \circ u(d)^* \end{aligned} \tag{55}$$

e.g., for $n = 2$,

$$\begin{aligned} n = 2: \quad e^{[x_0\mathbf{1}(2)+\vec{x}\vec{\sigma}]/2} &= e^{x_0/2}\left(\mathbf{1}(2)\cosh\frac{|\vec{x}|}{2} + \frac{\vec{\sigma}|\vec{x}|}{|\vec{x}|}\sinh\frac{|\vec{x}|}{2}\right) \\ &= e^{x_0/2}\begin{pmatrix} \cosh\frac{|\vec{x}|}{2} + \cos\theta\sinh\frac{|\vec{x}|}{2} & e^{-i\varphi}\sin\theta\sinh\frac{|\vec{x}|}{2} \\ e^{i\varphi}\sin\theta\sinh\frac{|\vec{x}|}{2} & \cosh\frac{|\vec{x}|}{2} - \cos\theta\sinh\frac{|\vec{x}|}{2} \end{pmatrix} \\ &= u(x)\begin{pmatrix} e^{(\vec{x}_0+|\vec{x}|)/2} & 0 \\ 0 & e^{(x_0-|\vec{x}|)/2} \end{pmatrix} u(x)^* \end{aligned} \tag{56}$$

For $n = 1$, one has an isomorphy for the abelian groups $\mathbf{D}(1) = e^{\mathbb{R}} \cong \mathbb{R}$.

In the general case, the diagonal matrix for a timespace point contains the n strictly positive spectral values

$$d \in \mathbf{D}(n): \operatorname{diag}(d) = \begin{pmatrix} e^{\xi_1} & 0 & \cdots & 0 \\ 0 & e^{\xi_2} & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & e^{\xi^n} \end{pmatrix}, \quad x_0 = \sum_{r=1}^{n} \xi_r \tag{57}$$

which realize n times the abelian causal group $\mathbf{D}(1)$. In a nontrivial boost submanifold $\mathbf{SD}(n)$, the group $\mathbf{D}(1)$ comes in the self-dual decomposable representation ('Prokrustes representation'), isomorphic to the orthochronous group $\mathbf{SO}^+(1, 1)$

$$\mathbf{D}(1) \cong \mathbf{SO}^+(1, 1) \ni d(\xi) = \begin{pmatrix} \cosh\frac{\xi}{2} & \sinh\frac{\xi}{2} \\ \sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} \end{pmatrix} \cong \begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix} \tag{58}$$

$$\left(-\frac{d^2}{d\xi^2} + 1\right) d(\xi) = 0$$

The unitary diagonalization transformation is determined up to the diagonal phases

$$u(d), u(x) \in \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \tag{59}$$

Therewith, timespace is isomorphic—as manifold—to a product of noncompact causal groups (Cartan subgroup) and a compact manifold

$$\begin{aligned} \mathbf{GL}(\mathbf{C}^n_\mathbf{R})/\mathbf{U}(n) &= \mathbf{D}(n) \cong \mathbf{D}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \\ \mathbf{SL}(\mathbf{C}^n_\mathbf{R})/\mathbf{SU}(n) &= \mathbf{SD}(n) \cong \mathbf{D}(1)^{n-1} \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \end{aligned} \tag{60}$$

and for the timespace translations

$$\log \mathbf{GL}(\mathbf{C}^n_\mathbf{R})/\log \mathbf{U}(n) \cong \mathbf{R}(n) \cong \mathbf{R}^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \tag{61}$$

The abelian group $\mathbf{D}(1)^n$ and its Lie algebra $\mathbf{R}^n$ constitute the Cartan skeleton of the timespace manifold $\mathbf{D}(n)$ and the timespace translations $\mathbf{R}(n)$, respectively.

The ranks (Helgason, 1978) n and $n - 1$ of the homogeneous manifold $\mathbf{D}(n)$ and $\mathbf{SD}(n)$, respectively, have to be seen in analogy to the rank of a Lie algebra or its Lie group, e.g., ranks n and $n - 1$ for $\mathbf{U}(n)$ and $\mathbf{SU}(n)$, respectively, with the manifold factorizations

$$\begin{aligned} \mathbf{U}(n) &\cong \mathbf{U}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \\ \mathbf{SU}(n) &\cong \mathbf{U}(1)^{n-1} \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1} \end{aligned} \tag{62}$$

2.4. Tangent Structure and Poincaré Group

A timespace $\mathbf{D}(n)$ and $\mathbf{R}(n)$ analysis is interpreted and performed with the linear forms (dual space[10]) $\mathbf{R}(n)^T$ of the timespace translations containing

[10] V^T denotes the dual vector space (linear forms) for a vector space V with the bilinear dual product $V^T \times V \to \mathbf{C}$, $(\omega, v) = \omega(v)$.

the weights (collection of eigenvalues). The linear forms will be called the *frequency (energy)* space for $n = 1$ and the *energy–momenta* space for $n \geq 2$. In the representation by the matrix algebra $\mathbb{C}(n)$ the 'double trace with one open slot' describes an isomorphism between translations and energy-momenta.

$$\begin{aligned}
&\mathbb{R}(n) \to \mathbb{R}(n)^T, \qquad q \mapsto \check{q} = \text{tr}\, q \circ \cdots \\
&\text{dual product: } \mathbb{R}(n)^T \times \mathbb{R}(n) \to \mathbb{R}, \qquad \langle \check{q}, x \rangle = \text{tr}\, q \circ x
\end{aligned} \tag{63}$$

With generalized Pauli matrices one has a dual bases

$$\begin{aligned}
\mathbb{R}(n)\text{-translations basis:} \quad & \{\rho(n)^j\}_{j=0}^{n^2-1} = \left\{\frac{\mathbf{1}(n)}{n}, \frac{\sigma(n)^a}{2}\right\}_{a=1}^{n^2-1} \\
\mathbb{R}(n)^T\text{-energy–momenta basis:} \quad & \{\check{\rho}(n)_j\}_{j=0}^{n^2-1} = \{\mathbf{1}(n), \sigma(n)^a\}_{a=1}^{n^2-1} \\
\text{dual bases:} \quad & \text{tr}\, \rho(n)^j \circ \check{\rho}(n)_k = \delta^j_k
\end{aligned} \tag{64}$$

The Cartan coordinates $\{\xi_r\}_{r=1}^n$ for the translations have their correspondence in *Cartan masses* $\{\mu_r\}_{r-1}^n$ for the energy-momenta, positive for positive energy-momenta

$$q \in \mathbb{R}(n)^T\text{:} \quad q = u(q)\begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & \mu_n \end{pmatrix} u(q)^\star \tag{65}$$

$$q \succcurlyeq 0 \Leftrightarrow \text{all } \mu_r \geq 0$$

The external group $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$ action on the causal manifold $\mathbf{D}(n)$ induces a faithful action of the adjoint external group Ad $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) \cong \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n$ on the tangent structures. This defines the *Poincaré group* as semidirect product (symbol $\times_s$) of the adjoint external group and the additive vector space groups

$$s \in \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})\text{:} \quad \begin{cases} \text{Ad}_\star s\text{:} \quad \mathbb{R}(n) \to \mathbb{R}(n), & \text{Ad}_\star s(x) = s \circ x \circ s^\star \\ \text{Ad}_\star \hat{s}\text{:} \quad \mathbb{R}(n)^T \to \mathbb{R}(n)^T, & \text{Ad}_\star s(q) = \hat{s} \circ q \circ s_\star \\ & \text{with } \hat{s} = s^{-1\star} \end{cases} \tag{66}$$

$$\mathbf{POIN}(n) = \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n \times_s \mathbb{R}(n)$$

In contrast to the direct product structure $\mathbf{D}(1) \times \mathbf{SD}(n)$ of the causal manifold, a decomposition of the timespace translations into time translations $\mathbb{T}$ and space translations $\mathbb{S}$ $(n - 1)$ or of the dual space into energy and momenta is incompatible with the action of $\mathbf{SL}(\mathbb{C}^n)/\mathbb{I}_n$ for $n \geq 2$, it is only compatible with the action of the adjoint compact subgroup $\mathbf{SU}(n)/\mathbb{I}_n$.

Two remarks are in order: In general, the adjoint action of a Lie group G on its Lie algebra log G can be characterized by $l \to \mathrm{Ad}g(l) = glg^{-1}$ leading to the semidirect product Ad $G \times_s T(G)$. The abelian normal subgroup $T(G)$ is the vector space structure of the Lie algebra log $G \cong T(G)$, i.e., a nonabelian Lie bracket has to be 'forgotten' (Bourbaki, 1989) for the translations $T(G)$.

In the case of a group with conjugation *, the involution $g \to \hat{g} = g^{-1*}$ is a group automorphism with the *-unitary group $U(G,*) = \{u \in G | u^* = u^{-1}\}$ as invariants. Correspondingly, the Lie algebra log G of a complex finite-dimensional Lie group with conjugation * is the direct sum of the isomorphic antisymmetric and symmetric real vector spaces log $G_\pm$ with $l_\pm^* = \pm l_\pm$. Here log G_- is the real Lie algebra of the unitary group $U(G, *)$. In order to be compatible with the conjugation $g \mapsto g^*$ and $l \mapsto l^*$, the adjoint action has to be modified to $l \mapsto \mathrm{Ad}_* g(l) = glg^*$. For the *-unitary subgroup $U(G, *)$, one has Ad $u = \mathrm{Ad}_* u$. Both subspaces log $G_\pm$ remain stable. The emerging semidirect adjoint group is $\mathrm{Ad}_*G \times_s \log G_+$ with the additive group structure of the vector space $\log G_+ \cong \log G/\log U(G, *)$. For all adjoint actions, the abelian center of the group is represented in the semidirect product via the additive structure of the translation factor.

For time alone, $n = 1$, the Poincaré group is the additive structure of the time translations

$$\mathbf{POIN}(1) = \mathbb{R}(1) = \log \mathbf{D}(1) \tag{67}$$

In the familiar relativistic case, $n = 2$, with

$$\begin{aligned} &\mathbf{POIN}(2) \cong \mathbf{SO}^+(1, 3) \times_s \mathbb{R}(2) \\ &\mathbb{R}(2) \cong \log \mathbf{D}(1) \oplus \log \mathbf{SO}^+(1, 3)/\log \mathbf{SO}(3) \end{aligned} \tag{68}$$

the Minkowski time-, light-, and spacelike translations are isomorphic to homogeneous manifolds with characteristic fixgroups ('little' groups) for the translations

$$\begin{aligned} &\mathbb{R}(2)^\pm_{\text{time}} && \cong \mathbf{D}(1) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}(3) && \cong \mathbf{GL}(\mathbb{C}^2_\mathbb{R})/\mathbf{U}(2) \\ &\mathbb{R}(2)^\pm_{\text{light}} && \cong \mathbf{SO}^+(1, 3)/\mathbf{SO}(2) \times_s \mathbb{R}^2 && \cong \mathbf{SL}(\mathbb{C}^2_\mathbb{R})/\mathbf{U}(1) \times_s \mathbb{C}_\mathbb{R} \\ &\mathbb{R}(2)_{\text{space}}\backslash\{0\} && \cong \mathbf{D}(1) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}^+(1, 2) && \cong \mathbf{GL}(\mathbb{C}^2_\mathbb{R})/\mathbf{U}(1, 1) \end{aligned} \tag{69}$$

The semidirect product $\mathbf{SO}(2) \times_s \mathbb{R}^2$ fixgroup of the lightlike translations is the Euclidean group.[11]

[11] For the Poincaré group $\mathbf{SO}^+(1, s) \times \mathbb{R}^{1+s}$ with $s \geq 1$ space dimensions the corresponding fixgroups are $\mathbf{SO}(s)$ (timelike), $\mathbf{SO}(s-1) \times_s \mathbb{R}^{s-1}$ (lightlike), and $\mathbf{SO}^+(1, s-1)$ (spacelike). The lightlike translations fixgroup for $s = 1$ is trivial $\{1\}$.

For the case $n \geq 2$, the disjoint decompositions of the spacelike and timelike manifolds into $(n - 1)$ submanifolds (Section 2.1) reflect different unitary fixgroups

$$n \geq 2: \begin{cases} \mathbb{R}(m, n - m)_{\text{space}} \cong \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(m, n - m) \\ \mathbb{R}(m, n - 1 - m)^{\pm}_{\text{light}} \cong \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(m, n - 1 - m) \times_s \mathbb{C}^{n-1}_{\mathbb{R}} \\ m = 1, \ldots, n - 1 \end{cases} \tag{70}$$

The fixgroup $\mathbf{U}(n)$ for the action of the external group $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n$ on the strictly positive translations $\mathbb{R}(n)^{\pm}_{\text{time}}$ should not be confused with the internal group $\mathbf{U}(n)$ which acts trivially (from the right) on timespace $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(n)$. A group G acting from the left on the subgroup classes $gU \in G/U$ has an U-isomorphic fixgroup for any point gU.

3. REPRESENTATIONS FOR TIMESPACE

The solution for an experimentally oriented formulation of a timespace dynamics requires an analysis, e.g., of a symmetry invariant, with respect to the operations used in the definition of a timespace manifold $\mathbf{D}(n) = \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(n)$. Starting from a purely algebraic framework, one can even say that an analysis with respect to time or timespace representations *introduces* the time or the timespace dependence of quantum mechanical operators (von Weizsäcker, 1993) or relativistic fields.

For a solution of a dynamics, the involved nondecomposable representations of the external-internal real Lie group

$$\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) \times \mathbf{U}(n) \tag{71}$$

have to be determined. The eigenvalue and eigenvector problems to be solved in a quantum structure is classically expressed by equations of motion. The eigenvalues (weights) for the action of a real Lie group are linear forms of the Cartan subalgebra and, therefore, have to be real. All weights (collection of eigenvalues) form a subgroup, discrete or continuous, in the additive group of the linear forms on the Lie algebra of the external-internal group.

The theorem that two diagonalizable finite-dimensional endomorphisms f, g are simultaneously diagonalizable, i.e., have a common eigenvector basis, if, and only if, they commute with each other $[f, g] = 0$, is the mathematical formalization of a central quantum operational structure. However, operators cannot be identified with states, mathematically: In general, endomorphisms f of a complex finite-dimensional space allow only a Jordan triangularization, i.e., no eigenvector basis. In general, the *nondecomposable* representation spaces of the external-internal group can be spanned by principal vectors.

Irreducible representations are a special case; their vector spaces can be spanned even by eigenvectors (diagonalization in the semisimple case).

For $n = 1$ with time as the causal group $\mathbf{D}(1)$, one has a dynamics for mass points (mechanics) where a Hamiltonian H representing–or defining—the time translations $\mathbb{R}(1)$ as generator of the causal group is analyzed with respect to the involved representations of time $\mathbf{D}(1)$, illustrated by the equation of motion $da/dt = [iH, a]$ for a quantum operator a, e.g., position X or momentum P. For example, the Hamiltonian for the N-dimensional isotropic harmonic oscillator $H = (P^2 + X^2)/2$ as $\mathbf{SU}(N)$-invariant or the hydrogen Hamiltonian $H = P^2/2 - 1/X$ as an invariant of the group $\mathbf{SU}(2)\times \mathbf{SU}(2)/\mathbb{I}_2 \cong \mathbf{SO}(4)$ (elliptic bound states) or $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbb{I}_2 \cong \mathbf{SO}^+(1, 3)$ (hyperbolic scattering states), both subsymmetries of $\mathbf{SU}(2, 2)/\mathbb{I}_2 \cong \mathbf{SO}(2, 4)$. The time dependence of quantum operators can be introduced (defined), e.g., by $a(t) = e^{iHt}ae^{-iHt}$.

For relativistic fields, $n = 2$, with the homogeneous timespace $\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(2)$ and the tangent Minkowski translations $\mathbb{R}(2)$, a dynamics analyzes an interaction with respect to the external-internal group, e.g., the analysis of invariant gauge vertices like $\overline{\mathbf{\Psi}}\ \gamma_k\ \mathbf{A}^k\ \mathbf{\Psi}$ with fermion and gauge degrees of freedom $\mathbf{\Psi}$ and $\mathbf{A}$, respectively in the standard model of elementary particles. Involving both noncompact and nonabelian structures, this much more complicated analysis works with representations of the external noncompact causality $\mathbf{D}(1)$ and Lorentz $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ properties as well as internal compact hypercharge $\mathbf{U}(1)$ and isospin $\mathbf{SU}(2)$ properties, as expressed by field equations, e.g., $\gamma_k\ \partial\mathbf{\Psi}/\partial x_k = i\mathcal{H}\ (\mathbf{\Psi}) = i\gamma_k\ \mathbf{A}^k\mathbf{\Psi}$.

3.1. Representations of the Causal Group

Any dynamics requires a *causal analysis* with respect to the group $\mathbf{D}(1) = \{e^\tau | \tau \in \mathbb{R}\}$.

All $\mathbf{D}(1)$ representations (Boerner, 1955; Saller, 1989) can be built from nondecomposable representations. The finite-dimensional, nondecomposable, unitary complex representations $e^\tau \mapsto (N|m)(\tau)$ of the real Lie group $\mathbf{D}(1)$ are characterized by positive integers $N \in \mathbb{N}$ for the dimension $1 + N$ of the representation space and a real number m—a frequency (energy) for time and a mass for timespace—from the dual group, the linear frequency (mass) space $\log \mathbf{D}(1)^T \cong \mathbb{R}$. They involve a power $1 + N$ *nilpotent* part Λ_N (nil-Hamiltonian) nontrivial for $N \neq 0$,

$$\mathbf{D}(1) \ni e^\tau \mapsto (N|m)(\tau) = e^{i\Lambda_N\tau}e^{im\tau} \qquad \text{with} \begin{cases} N = 0, 1, 2, \ldots \\ (\Lambda_N)^N \neq 0, \quad N \neq 0 \\ (\Lambda_N)^{1+N} = 0 \end{cases} \tag{72}$$

$$(N|m)(\tau) = e^{\Lambda_N d/dm}e^{im\tau}, \qquad \operatorname{tr}(N|m)(\tau) = (1 + N)e^{im\tau}$$

The following explicit examples with nilcyclic matrices Λ_N illustrate the abstract structure:

$$
\begin{aligned}
(0|m)(\tau) &= e^{im\tau}\\
(1|m)(\tau) &\cong \begin{pmatrix} 1 & i\lambda\tau \\ 0 & 1 \end{pmatrix} e^{im\tau} = \begin{pmatrix} 1 & \lambda\, d/dm \\ 0 & 1 \end{pmatrix} e^{im\tau} \qquad (73)\\
(2|m)(\tau) &\cong \begin{pmatrix} 1 & i\lambda\tau & (i\lambda\tau)^2/2! \\ 0 & 1 & i\lambda\tau \\ 0 & 0 & 1 \end{pmatrix} e^{im\tau} = \begin{pmatrix} 1 & \lambda\, d/dm & (\lambda^2/2!)\, d^2/dm^2 \\ 0 & 1 & \lambda\, d/dm \\ 0 & 0 & 1 \end{pmatrix} e^{im\tau}\\
\Lambda_0 &= 0, \qquad \Lambda_1 \cong \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_2 \cong \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

To take care of the *real* structure of the causal group $\mathbf{D}(1)$, a *complex* $\mathbf{D}(1)$-representation has to be a 1-dimensional subgroup of a *unitary group*. Therewith the constant m has to be real in nondecomposable representations. The $\mathbf{D}(1)$-images for the 1-dimensional representations $(0|m)$ for $m \neq 0$ (harmonic oscillator) are isomorphic to $\mathbf{U}(1)$. The representations $(1|m)$ (e.g., a free mass point for $m = 0$) have real 1-dimensional faithful images in the *indefinite unitary* group $\mathbf{U}(1, 1)$, $(2|m)$ in $\mathbf{U}(2, 1)$, $(3|m)$ in $\mathbf{U}(2, 2)$, etc. We have

$$
(N|m) \text{ in } \mathbf{U}(N_+, N_-) \quad \text{with} \begin{cases} N_+ + N_- = 1 + N \\ N_+ - N_- = \begin{cases} 1, & N = 0, 2, \ldots \\ 0, & N = 1, 3, \ldots \end{cases} \end{cases} \qquad (74)
$$

In addition to the discrete dimension $1 + N$, the nondecomposable causal representations involve two continuous constants, m and λ: Only the frequency (mass) m is an invariant of the causal group $\mathbf{D}(1)$. It is the *causal unit of the representation.* The matrix form of the nilpotent Λ_N and the real constant[12] $\lambda \neq 0$ are determined up to equivalence $g\Lambda_N g^{-1}$.

Representations with opposite frequency (mass) $(N|\pm m)$ are dual to each other and nonequivalent for $m^2 > 0$. The self-dual, m^2-dependent formulation contains two dual representations, e.g., in $\mathbf{SO}(2)$ for $N = 0$

$$
\begin{aligned}
&(0|m)(\tau) \oplus (0| - m)(\tau)\\
&\quad = (0\|m^2)(\tau) \cong \begin{pmatrix} e^{im\tau} & 0 \\ 0 & e^{-im\tau} \end{pmatrix} \cong \begin{pmatrix} \cos m\tau & (i/m) \sin m\tau \\ im \sin m\tau & \cos m\tau \end{pmatrix} \qquad (75)
\end{aligned}
$$

[12] E.g., for $N = 1$ with $\Lambda_1 = \lambda\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ the constant λ is transformed with $g = \left(\begin{smallmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{smallmatrix}\right)$ to $\lambda \mapsto e^{2\alpha}\lambda$. A similar structure (Becchi *et al*, 1976; Saller, 1991) is used for the 'gauge-fixing constant' in quantum gauge theories. The gauge-fixing constant has to be nontrivial—its value is physically irrelevant. The gauge structure in the BRS formulation is nilpotent.

Only the positive unitary 1-dimensional representations $(0|m)$ are irreducible; they are unfaithful—the $\mathbf{U}(1)$-represented time is periodic. The indefinite unitary faithful representations $(N|m)$ for $N \geq 1$ are reducible, but nondecomposable. The lowest dimensional faithful $\mathbf{D}(1)$ representations $(1|m)$ will be called *fundamental.*

The pairs $(N|m)$ with a positive integer (dimension) and a real number (causal unit) form the abelian *representation monoid of the causal group* for all equivalence classes of *nondecomposable* causal representations

$$\begin{aligned} \text{mon } \mathbf{D}(1) &= \{(N|m)|N = 0, 1, \ldots, m \in \mathbb{R}\} \cong \mathbb{N} \times \mathbb{R} \\ &(N_1|m_1) + (N_2|m_2) = (N_1 + N_2|m_1 + m_2) \\ &\text{neutral element: } (0|0) \end{aligned} \tag{76}$$

The *weight group* of $\mathbf{D}(1)$ is the regular subgroup of the representation monoid, it characterizes the *irreducible* representations of the causal group and is the dual space $\log \mathbf{D}(1)^T$ of the causal translations $\log \mathbf{D}(1)$

$$\begin{aligned} \text{grp } \mathbf{D}(1) &= \{(0|m)|m \in \mathbb{R}\} \cong \mathbb{R} \\ \text{mon } \mathbf{D}(1) &\supset \text{grp } \mathbf{D}(1) \end{aligned} \tag{77}$$

The $\mathbf{U}(1)$-*weights* (oriented winding numbers) for the compact quotient group $\mathbf{U}(1) \cong \mathbf{D}(1)/e^z$ form a discrete subgroup, the *representation group for* $\mathbf{U}(1)$

$$\text{grp } \mathbf{U}(1) = \{(0|Z)|Z \in \mathbb{Z}\} = \mathbb{Z} \tag{78}$$

3.2. Representations of $\mathbf{GL}(\mathbb{C}_\mathbb{R})$

All *irreducible* complex representation of the real abelian Lie group $\mathbf{GL}(\mathbb{C}_\mathbb{R}) = \mathbf{D}(1) \times \mathbf{U}(1)$ are 1-dimensional and characterized by an integer winding number [$\mathbf{U}(1)$- weight] and a complex number

$$\begin{aligned} &\mathbf{GL}(\mathbb{C}_\mathbb{R}) \ni \delta = e^{\tau + i\alpha} \mapsto |\delta|^{im}\left(\frac{\delta}{\bar{\delta}}\right)^{z/2} = \delta^{(im+z)/2}\bar{\delta}^{(im-z)/2} = e^{im\tau}e^{Zi\alpha} \\ &(m; Z) \in \mathbb{C} \times \mathbb{Z} \end{aligned} \tag{79}$$

Complex representations of real groups have to be unitary. Unitary irreducible $\mathbf{D}(1)$-representations are necessarily positive unitary. All unitary irreducible representations have a real causal unit m [$\mathbf{D}(1)$-weight]

$$\text{grp } \mathbf{GL}(\mathbb{C}_\mathbb{R}) = \text{grp } \mathbf{D}(1) \times \text{grp } \mathbf{U}(1) = \{(m; Z)\} \cong \mathbb{R} \times \mathbb{Z} \tag{80}$$

As seen in the previous subsection, the group of the $\mathbf{GL}(\mathbb{C}_\mathbb{R})$-*weights* is the regular group of the $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})$-*representation monoid* for all equivalence classes of the unitary *nondecomposable* representations

$$\begin{aligned}\text{mon } \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) &= \text{mon } \mathbf{D}(1) \times \text{grp } \mathbf{U}(1) = \{(N|m; Z)\}\\ &\cong \mathbb{N} \times \mathbb{R} \times \mathbb{Z} \\ \text{mon } \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) &\supset \text{grp } \mathbf{GL}(\mathbb{C}_{\mathbb{R}})\end{aligned} \tag{81}$$

3.3. Finite-Dimensional Representations of $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$

All complex representations of the compact real $(n^2 - 1)$-dimensional Lie group $\mathbf{SU}(n)$, $n \geq 2$, are decomposable into irreducible ones which have finite dimensions. The irreducible $\mathbf{SU}(n)$-representations are characterized—according to rank $(n - 1)$ and Cartan subgroup $\mathbf{U}(1)^{n-1}$—by $(n - 1)$ positive integers with the additive $\mathbf{SU}(n)$-*representation monoid* for the equivalence classes

$$\text{mon } \mathbf{SU}(n) = \{[2J_1, \ldots, 2J_{n-1}] | 2J_k = 0, 1, \ldots\} \cong \mathbb{N}^{n-1} \tag{82}$$

The positive integers reflect the $\mathbb{N}$-linear combination of the $(n - 1)$ *fundamental* representations $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$, e.g., the Pauli spinor representation $[1]$ for $\mathbf{SU}(2)$ or the quark and antiquark representations $[1, 0]$ and $[0, 1]$ for $\mathbf{SU}(3)$. The adjoint representation $[1, 0, \ldots, 0, 1]$ is faithful only for the adjoint group $\mathbf{SU}(n)/\mathbb{I}_n$, e.g., the adjoint $\mathbf{SU}(2)$-representation $[2]$ for $\mathbf{SO}(3)$ or the $\mathbf{SU}(3)$-octet representation $[1, 1]$ for $\mathbf{SU}(3)/\mathbb{I}_3$.

The $\mathbf{SU}(n)$-representation monoid is the positive cone (dominant weights) in all $\mathbf{SU}(n)$-*weights* which constitute a discrete subgroup in the linear forms $\log \mathbf{SU}(n)^T$ of the Lie algebra

$$\begin{aligned}\text{grp } \mathbf{SU}(n) &= \{[2j_1, \ldots, 2j_{n-1}] | 2j_k \in \mathbb{Z}\}\\ &= \text{grp } \mathbf{U}(1)^{n-1} \cong \mathbb{Z}^{n-1} \\ \text{mon } \mathbf{SU}(n) &\subset \text{grp } \mathbf{SU}(n)\end{aligned} \tag{83}$$

The integers $\{2j_r\}_{r=1}^{n-1}$ can be related to the winding numbers of the Cartan $\mathbf{U}(1)$'s involved. Since $\mathbf{SU}(n)$ are simple groups, they come in self-dual $\mathbf{SO}(2)$ representations. Especially for $\mathbf{SU}(2)$ the half-integers $(J, j) \in \mathbb{N}/2 \times \mathbb{Z}/2$ are called spin and its third component.

The *defining* representation with a complex n-dimensional representation space can be written with the generalized Pauli (Section 2.1) matrices

$$[1, 0, \ldots, 0](\alpha) \cong e^{\alpha_a i\sigma(n)^a/2} \tag{84}$$

where the Cartan subgroup $\mathbf{U}(1)^{n-1}$ is represented with the $(n-1)$ diagonal matrices

$$\mathbf{U}(1)^{n-1} = \left\{ \exp \sum_{k=2}^{n} \alpha_{k^2-1} \frac{i\sigma(n)^{k^2-1}}{2} \middle| \alpha_{k^2-1} \in \mathbb{R} \right\} \tag{85}$$

Taking together the diagonals of the $(n-1)$ matrices $\{\frac{1}{2}\sigma(n)^{k^2-1}\}_{k=2}^{n}$, one obtains the n weights $\{w_r\}_{r=1}^{n}$ of the defining $\mathbf{SU}(n)$-representation in the real $(n-1)$-dimensional weight space. In the normalization[13] with the Pauli matrices, the defining weights occupy the corners of a regular fundamental simplex, centered at the origin, as expressed by the $[(n-1) \times n]$-matrix

$$\text{weights } [1, 0, \ldots, 0] \cong \text{simplex } (n)$$

$$= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \cdots \\ \cdots \\ w_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\ -1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\ 0 & -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\ 0 & 0 & -\frac{3}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\ & \cdots & & \cdots & \cdots \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{n-1}{\sqrt{\binom{n}{2}}} \end{pmatrix},$$

$$\sum_{r=1}^{n} w_r = 0$$

$$\|w_k - w_j\| = \delta_{kj} \tag{86}$$

$$\|w_k\| = \sqrt{\frac{n-1}{2n}}$$

[13]The integer winding number normalization arises with the basis $\{\sqrt{\binom{k}{2}}\sigma(n)^{k^2-1}\}_{k=2}^{n}$.

All complex *finite-dimensional* irreducible representations of the simple Lie group $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$, $n \geq 2$, are characterized—according to the Cartan subgroup $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})^{n-1}$—by $2(n-1)$ positive integers, interpretable as $(n-1)$ 'left' and $(n-1)$ 'right' winding numbers

$$\begin{aligned} \times\text{mon}_{\text{fin}}\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}}) &= \{[2L_1, \ldots, 2L_{n-1}|2R_{n-1}, \ldots, 2R_1]|2L_k, 2R_k \in \mathbb{N}\} \\ &= \text{mon } \mathbf{SU}(n) \times \text{mon } \mathbf{SU}(n) \cong \mathbb{N}^{n-1} \times \mathbb{N}^{n-1} \end{aligned} \tag{87}$$

reflecting the $\mathbb{N}$-linear combinations from $2(n-1)$ *fundamental* representations (one 1, elsewhere 0). Also, the weight group is the 'square' of the weight group for the unitary subgroup

$$\begin{aligned} \times\text{grp}_{\text{fin}}\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}}) &= \{[2l_1, \ldots, 2l_{n-1}|2r_{n-1}, \ldots, 2r_1]|2l_k, 2r_k \in \mathbb{Z}\} \\ &= \text{grp } \mathbf{SU}(n) \times \text{grp } \mathbf{SU}(n) \cong \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \end{aligned} \tag{88}$$

$$\times\text{mon}_{\text{fin}}\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}}) \subset \times\text{gr}_{\text{fin}}\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$$

The finite-dimensional irreducible $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-representations are not necessarily unitary. However, the monoid and the group allow a *conjugation*

$$\begin{aligned} [2L_1, \ldots, 2L_{n-1}|2R_{n-1}, \ldots, 2R_1]^\times &= [2R_1, \ldots, 2R_{n-1}|2L_{n-1}, \ldots, 2L_1] \\ [2l_1, \ldots, 2l_{n-1}|2r_{n-1}, \ldots, 2r_1]^\times &= [2r_1, \ldots, 2r_{n-1}|2l_{n-1}, \ldots, 2l_1] \end{aligned} \tag{89}$$

The equivalence classes with respect to this conjugation characterize the equivalence classes of the finite-dimensional irreducible representations of the *complex* group $\mathbf{SL}(\mathbb{C}^n)$.

The conjugated pair of the two *defining* finite-dimensional $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-representations uses the Pauli matrices, e.g., for $n = 2$ the left- and right-handed Wely representations $[1|0]$ and $[0|1]$

$$\begin{aligned} [1, 0, \ldots, 0|0, \ldots, 0, 0](x, \alpha) &\cong e^{(x_a + i\alpha_a)\sigma(n)^a/2} \\ [0, 0, \ldots, 0|0, \ldots, 0, 1](x, \alpha) &\cong e^{-(x_a + i\alpha_a)\sigma(n)^a/2} \end{aligned} \tag{90}$$

These representations are not unitary, they are equivalent for the complex group $\mathbf{SL}(\mathbb{C}^n)$.

Only the self-conjugated irreducible representations are also unitary (indefinite unitary). They define the representation monoid and weight group for the adjoint group $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n$:

$$\begin{aligned} \text{mon } \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n &= \{[2J_1, \ldots, 2J_{n-1}|2J_{n-1}, \ldots, 2J_1]|2J_k \in \mathbb{N}\} \\ &\cong \mathbb{N}^{n-1} \end{aligned} \tag{91}$$

$$\text{grp } \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n \cong \mathbb{Z}^{n-1}$$

with the $(n-1)$ *fundamental* representations $[1, 0, \ldots, 0|0, \ldots, 0, 1]$, etc.

The conjugation-compatible analogue to the real $(n^2 - 1)$-dimensional adjoint representation for $\mathbf{SU}(n)$ is the irreducible unitary representation of $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$ on a real n^2-dimensional space, e.g., on the timespace translations $\mathbb{R}(n) \cong \log \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\log \mathbf{U}(n)$, faithful for $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n$, i.e., in the case $n = 2$ for the Lorentz group $[\mathbf{SO}^+(1,3)]$

$$\begin{aligned} \times\text{-adjoint representation:} \quad & [,1, 0, \ldots, 0|0, \ldots, 0, 1] \\ n = 2: \quad & [1|1] \end{aligned} \tag{92}$$

The n^2-dimensional indefinite unitary irreducible $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-representation is called the *defining* representation of the adjoint group $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}_n$, e.g., the Minkowski representation of $\mathbf{SO}^+(1,3)$ on $\mathbb{R}(2)$.

Any representation of $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$ is a representation for $\mathbf{SU}(n)$—in general decomposable—and gives—by the quotient of the represented groups—a realization of the Sylvester manifold $\mathbf{SD}(n) = \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{SU}(n)$, e.g., for $n = 2$ with the Pauli matrices $\vec{\sigma}$ and the Lorentz boost matrices B

$$[1|0](x) = e^{\vec{x}\vec{\sigma}/2}, \qquad [0|1](x) = e^{-\vec{x}\vec{\sigma}/2}$$

$$[1|1](x) = e^{\vec{x}\vec{B}}, \qquad \vec{x}\vec{B} = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{pmatrix} \tag{93}$$

3.4. Irreducible Representations of $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$

As shown by Gel'fand and Neumark (1957; Gel'fand *et al.*, 1966; Neumark, 1963), all irreducible $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-representations can be characterized by the irreducible representations of the Cartan subgroup $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})^{n-1}$.

To illustrate the case $n = 2$: The representation spaces of

$$\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) = \{\lambda = e^{(\vec{x}+i\vec{\alpha})\vec{\sigma}/2} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} | \det \lambda = 1\} \tag{94}$$

are subspaces of the complex vector space $\mathbb{C}^2 = \{f\colon \mathbb{C}^2 \to \mathbb{C}\}$ with the complex-valued mappings on the vector space $\mathbb{C}^2$. The $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$-action is induced by the defining representation on $\mathbb{C}^2$

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\lambda} & \mathbb{C}^2 \\ f\downarrow & & \downarrow {}_\lambda f, \\ \mathbb{C} & \xrightarrow[\mathrm{id}_{\mathbb{C}}]{} & \mathbb{C} \end{array} \qquad {}_\lambda f(z_1, z_2) = f(\lambda^{-1} \bullet (z_1, z_2)) \tag{95}$$

$$\lambda^{-1} \bullet (z_1, z_2) = (z_1, z_2)\lambda = (z_1, z_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2)$$

$${}_{\lambda' \circ \lambda} f = {}_{\lambda'}({}_\lambda f)$$

The irreducible $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)$-representations use the irreducible representations of its Cartan subgroup $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})$ on the subspace of the mappings which are $(\nu_1 - 1|\nu_2 - 1)$ homogeneous with respect to a 'relative' $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})$ for the two components $z_{1,2}$

$$\mathbb{C}^{\mathfrak{a}2}_{(\nu_1|\nu_2)} = \{f \in \mathbb{C}^{\mathfrak{a}2}|f(\delta z_1, \delta z_2) = \delta^{\nu_1-1}\bar{\delta}^{\nu_2-1}f(z_1, z_2) \text{ for } \delta \in \mathbb{C}\}$$

$$\text{with} \quad \begin{cases} \nu_1 = \dfrac{im + Z}{2}, \qquad \nu_2 = \dfrac{im - Z}{2} \\ (m; Z) = (-i(\nu_1 + \nu_2); \nu_1 - \nu_2) \in \mathbb{C} \times \mathbb{Z} \end{cases} \tag{96}$$

Those functions involve an integer winding number $\pm Z$ (spin $Z/2$) for the 'relative' phase group $\mathbf{U}(1) \subset \mathbf{SU}(2)$ and a complex number m for the 'relative' causal group $\mathbf{D}(1) \subset \mathbf{SD}(2) = \mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{SU}(2)$. Since the $(\nu_1 - 1|\nu_2 - 1)$ homogeneous mappings have the orbit properties

$$f(z_1, z_2) = z_2^{\nu_1-1}\bar{z}_2^{\nu_2-1} f\left(\frac{z_1}{z_2}, 1\right) \tag{97}$$

the group $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)$ acts on the corresponding vector space

$$F(\zeta) = f(\zeta, 1) \in \mathbb{C}^{\mathfrak{a}} = \{F: \mathbb{C} \to \mathbb{C}\} \tag{98}$$

in the following form:

$$\lambda \mapsto D^{(m;Z)}(\lambda), \qquad F \mapsto D^{(m;Z)}(\lambda)(F) = {}_{\lambda}F$$

$$\begin{aligned} {}_{\lambda}F(\zeta) &= F\left(\frac{\alpha\zeta + \gamma}{\beta\zeta + \delta}\right)|\beta\zeta + \delta|^{im}\left(\frac{\beta\zeta + \delta}{\overline{\beta\zeta + \delta}}\right)^{Z/2} \\ &= F\left(\frac{\alpha\zeta + \gamma}{\beta\zeta + \delta}\right)(\beta\zeta + \delta)^{\nu_1-1}(\overline{\beta\zeta + \delta})^{\nu_2-1} \end{aligned} \tag{99}$$

The pairs $(m;Z) \in \mathbb{C} \times \mathbb{Z}$ characterize all irreducible complex representations of $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)$, not necessarily unitary. The *finite-dimensional irreducible* $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)$-representations of the previous subsection arise with an integer imaginary 'causal number' m

$$\begin{aligned} &(m; Z) \in i\mathbb{Z} \times \mathbb{Z} \\ &\quad \times \text{grp}_{\text{fin}}\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2) = \{[\nu_1| - \nu_2] = [2l|2r]\} \\ &\quad = \text{grp}\,\mathbf{SU}(2) \times \text{grp}\,\mathbf{SU}(2) = \mathbb{Z} \times \mathbb{Z} \end{aligned} \tag{100}$$

As to be expected from the abelian group $\mathbf{GL}(C_R)$, the *unitary principal irreducible* $\mathbf{SL}(C_R^2)$-representations are characterized by a real *causal unit m (mass)*

$$\begin{aligned} \text{grp}_{\text{princ}}\mathbf{SL}(C_R^2) &= \{(m; Z)\} \\ &= \text{grp}\ \mathbf{GL}(C_R) \cong \mathbf{R} \times \mathbb{Z} \end{aligned} \tag{101}$$

which reflects the $\mathbf{U}(2)$-unitary representations (conjugation *) of the Cartan subgroup

$$\begin{aligned} &e^{(x_3+i\alpha_3)\sigma^3/2} \mapsto u = e^{(imx_3+iZ\alpha_3)\sigma^3/2} \\ &(m; Z) \in \mathbf{R} \times \mathbb{Z} \Rightarrow u^\star = u^{-1} \in \mathbf{U}(1)_3 \subset \mathbf{SU}(2) \end{aligned} \tag{102}$$

The unitary representations with trivial causal unit $m = 0$ are the finite-dimensional self-conjugated representations of the previous subsection

$$(0; 4J) \cong [2J|2J] \tag{103}$$

Those massless indefinite unitary irreducible representations with the defining 4-dimensional Minkowski representation $[2J|2J] = 2J[1|1]$ are used for gauge fields in relativistic field theories.

With respect to the indefinite unitary group $\mathbf{U}(1,1)$ with conjugation $\times$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar\delta & \bar\beta \\ \bar\gamma & \bar\alpha \end{pmatrix} \tag{104}$$

diagonal representations of the Cartan subgroup in $\mathbf{SU}(1, 1)$ have to be of the form

$$(m; Z) = (i\rho; 0) \in i\mathbf{R} \Rightarrow u = e^{-\rho x_3\sigma^3/2} \in \mathbf{SU}(1, 1) \tag{105}$$

leading to the *unitary supplementary irreducible* $\mathbf{SL}(C_R^2)$-representations with trivial winding numbers and an imaginary causal number $i\rho$

$$\text{grp}_{\text{suppl}}\mathbf{SL}(C_R^2) = \{(i\rho; 0)\} = i\mathbf{R} \tag{106}$$

The equivalences classes for the representations in the principal and supplementary series are given in Neumark (1963), Gelfand and Neumark (1957), and Gel'fand *et al.* (1966). For the *positive unitary* representations[14] one has to discuss also the scalar products for the representations spaces, especially for the supplementary series.

The generalizations for $\mathbf{SL}(C_R^n)$, $n \geq 2$, with Cartan subgroup $\mathbf{GL}(C_R)^{n-1}$, are given for the principal representations with $(n - 1)$ real causal units and $(n - 1)$ integer winding numbers

[14] Already Gel'fand and Neumark (1957) call the requirement of positive unitarity for $\mathbf{SL}(C_R^n)$-representations 'in a certain sense unnatural'.

$$\text{grp}_{\text{princ}}\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}}) = \{(m_1, \ldots, m_{n-1}; Z_1, \ldots, Z_{n-1})\}$$
$$= \text{grp } \mathbf{GL}(\mathbb{C}_{\mathbb{R}})^{n-1} \cong \mathbb{R}^{n-1} \times \mathbb{Z}^{n-1} \quad (107)$$
$$(0, \ldots, 0; 2J_1, \ldots, 2J_{n-1}) \cong [2J_1, \ldots, 2J_{n-1}|2J_{n-1}, \ldots, 2J_1]$$

The supplementary series has to take into account the diagonal [**U**(1, 1)] structure

$$\begin{pmatrix} e^{imx+iZ\alpha} & 0 \\ 0 & e^{+i\overline{m}x+iZ\alpha} \end{pmatrix} \in \mathbf{U}(1, 1) \qquad \text{for} \quad (m; Z) \in \mathbb{C} \times \mathbb{Z} \quad (108)$$

Therewith the supplementary weights are characterized by coinciding winding number pairs and conjugated causal numbers [more details are given in Neumark (1963), Gelfand and Neumark (1957), and Gel'fand *et al.* (1966)]

$$(m_1, \ldots, m_{n-1}; Z_1, \ldots, Z_{n-1}) \in \mathbb{C}^{n-1} \times \mathbb{Z}^{n-1}$$
$$\text{with entries } (m, \overline{m}; Z, Z) \quad (109)$$

4. ENERGY-MOMENTA MEASURES

As to be expected from their Cartan subgroups $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})^{n-1}$, $n \geq 2$, also the groups $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$ have *reducible, but nondecomposable* representations as first discussed by Shelobenko (1958, 1959).

Therewith, I suspect[15] that the $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$*-representation monoid with real causal units* for the equivalence classes of all unitary nondecomposable representations is given by the representation monoid for the Cartan subgroup

$$\text{mon}_{\text{princ}}\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) \overset{?}{=} \{(N_1, \ldots, N_n|m_1, \ldots, m_n; Z_1, \ldots, Z_n)\}$$
$$= \text{mon } \mathbf{GL}(\mathbb{C}_{\mathbb{R}})^n \cong \mathbb{N}^n \times \mathbb{R}^n \times \mathbb{Z}^n \quad (110)$$

One has for the weight groups with real causal units[16]

$$\text{grp}_{\text{princ}}\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) = \{(m_1, \ldots, m_n; Z_1, \ldots, Z_n)\}$$
$$= \text{grp } \mathbf{GL}(\mathbb{C}_{\mathbb{R}})^n \cong \mathbb{R}^n \times \mathbb{Z}^n$$
$$\text{grp } \mathbf{U}(n) = \{[Z_1, \ldots, Z_n]\}$$
$$= \text{grp } \mathbf{U}(1)^n \cong \mathbb{Z}^n \quad (111)$$

[15] A mathematically rigorous classification of all *nondecomposable unitary* representations of $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$ for $n \geq 2$ would be appreciated.

[16] With $\mathbf{U}(n) \cong \dfrac{\mathbf{U}(1) \times \mathbf{SU}(n)}{\mathbb{I}_n}$, one has to take care of the phase correlations for both unitary factors, e.g., relevant for the isospin-hypercharge correlation in the standard model (Saller, 1992b).

If such a conjecture is true, one is led to the suggestion for the harmonic analysis of the causal timespace manifolds

$$\begin{aligned} \text{mon}_{\text{princ}}\mathbf{D}(n) &\stackrel{?}{=} \{(N_1, \ldots, N_n|m_1, \ldots, m_n)\} \\ &= \text{mon } \mathbf{D}(1)^n = \mathbb{N}^n \times \mathbb{R}^n \end{aligned} \tag{112}$$

i.e., the nondecomposable realizations of the homogeneous spaces $\mathbf{D}(n)$ would be characterized by n natural numbers $N_k \in \mathbb{N}$ for the dimensions (discrete invariants) and n causal units $m_k \in \mathbb{R}$ (continuous invariants).

In this section, I shall try to make those structures concrete, following the analogies to the abelian case used for the Cartan subgroups.

4.1. Algebra of Causal Measures

Since representations of the causal group are characterized by a continuous real invariant m (mass), it is appropriate to use a *Lebesque measure* on the Lie algebra linear forms. The causal representations $(N|m)$ are expressible with Dirac distributions (point measures) $\delta_m \cong \delta(m - \mu)$ and their derivatives, e.g.,

$$\begin{aligned} (0|m)(\tau) &= \int d\mu\, \delta(m-\mu)e^{i\mu\tau} \\ (1|m)(\tau) &= \int d\mu \begin{pmatrix} \delta(m-\mu) & \lambda\delta'(m-\mu) \\ 0 & \delta(m-\mu) \end{pmatrix} e^{i\mu\tau} \\ (2|m)(\tau) &= \int d\mu \begin{pmatrix} \delta(m-\mu) & \lambda\delta'(m-\mu) & \frac{\lambda^2}{2!}\delta''(m-\mu) \\ 0 & \delta(m-\mu) & \lambda\delta'(m-\mu) \\ 0 & 0 & \delta(m-\mu) \end{pmatrix} e^{i\mu\tau} \end{aligned} \tag{113}$$

etc.

A *causal measure* of the mass space $\log \mathbf{D}(1)^T \cong \mathbb{R}$ for the analytic manifold $\mathbf{D}(1)$ is defined by its property to define a function, analytic in the causal translations $\tau \in \mathbb{R}$

$$e^\tau \mapsto \int d\mu\, h(\mu)e^{i\mu\tau} \tag{114}$$

A measure can be multiplied with a complex number. Two measures can be added and multiplied via the δ-additive convolution, induced by the composition in the representation monoid

$$(h * h')(\mu) = \int d\mu_1\, d\mu_2\, h(\mu_1)\delta(\mu_1 + \mu_2 - \mu)h'(\mu_2) \tag{115}$$

Therewith the causal measures means $\mathbb{R}$ have the structure of an *abelian unital algebra* with the unit given by the underived Dirac measure for trivial frequency (mass) δ_0.

A causal measure h_N has the *momentum* $N \in \mathbb{N}$ if it obeys the conditions

$$N = 0: \quad \text{meas}_0\, \mathbb{R} = \left\{ h_0 \middle| \int d\mu\; h_0(\mu) \neq 0 \right\} \tag{116}$$

$$N \geq 1: \quad \text{meas}_N\, \mathbb{R} = \left\{ h_N \middle| \begin{cases} \int d\mu\; \mu^k h_N(\mu) = 0, & k = 0, \ldots, N-1 \\ \int d\mu\; \mu^N h_N(\mu) \neq 0 \end{cases} \right\}$$

with the Dirac point measures as examples

$$\delta^{(N)}(m - \mu) \cong \delta_m^{(N)} \in \text{meas}_N\, \mathbb{R} \tag{117}$$

The normalization $\int d\mu\; h(\mu)$ of a causal measure reflects the representation of the causal group unit $1 \in \mathbf{D}(1)$. Its first causal momentum will be called

$$\textit{causal unit:} \quad m = \int d\mu\; \mu h(\mu) \tag{118}$$

With respect to the possibly indefinite unitary causal representations, e.g., $(1|m)$ in $\mathbf{U}(1, 1)$, the measures are not required to be positive definite. This feature has to be taken care of in the probability interpretation of quantum theories. It will be discussed in connection with the spacelike supported parts of a propagator, i.e., with respect to the in- and outgoing particle interpretable causal representations (Section 5.3).

The functions arising in the quantization of linear fields (Section 1.2) have the $\mathbf{D}(1)$-analysis

$$\begin{aligned} \mathscr{E}_n(\tau^2) &= \int d\mu\; h_0^{(n)}(\mu) e^{i\mu\tau} \\ h_0^{(n)}(\mu) &= \frac{1}{B\left(\frac{1}{2}, \frac{1}{2} + n\right)}\, \vartheta(1 - \mu^2)(1 - \mu^2)^{n - 1/2}, \qquad \int d\mu\; h_0^{(n)}(\mu) = 1 \end{aligned} \tag{119}$$

Any dynamics determines a subalgebra of the causal algebra, e.g.: The subalgebra $\text{meas}_0\, \mathbb{R}$ is related to the measures for irreducible causal representations. Its subalgebra $\log \mathbf{D}(1)^T \cong \mathbb{R}$ uses only the point supported Dirac measures $\log \mathbf{D}(1)^T \cong \{\delta_m | m \in \mathbb{R}\}$—it is the algebra for the irreducible $\mathbf{U}(1)$-representations of the causal group (positive unitary characters). Its discrete subgroup with the integers $\{\delta_{zm} | z \in \mathbb{Z}\} \cong \mathbb{Z}$ is used for the time development of a harmonic oscillator with frequency m. Any subset of the causal measure algebra together with the unit generates a subalgebra. Such a set may be used in a perturbative approach to generate the full subalgebra, associated to the dynamics to be solved. For example, the 2-elementic set $\{\delta_{\pm m} | m \neq 0\}$, associated to the irreducible representations $e^{\pm im\tau}$, generates the causal subgroup $\mathbb{Z}$ and the associated representations for the quantum oscillator.

4.2. Representation Structure of Linear Fields

Linear field theories on the 4-dimensional timespace $\mathbf{R}(2)$ (Section 1.2) are not compatible with the algebra structure of causal measures. Fields with distributive quantization cannot be used as a generating set. The divergencies in Feynman integrals, e.g., in the vacuum polarization of quantum electrodynamics, involving the undefined product $[\gamma_k \mathbf{c}^k(m|x) + i\mathbf{s}(m|x)]^2$, show that the convolution product does not make sense for linear fields.

In the distributive quantization of quantum fields (Section 1.2), one uses the naive analogue of the 1-dimensional time $\mathbf{D}(1)$ structure

$$(0||m^2)(t) \cong \begin{pmatrix} \cos mt & \frac{i}{m} \sin mt \\ im \sin mt & \cos mt \end{pmatrix} \tag{120}$$
$$\begin{pmatrix} \cos mt \\ i \sin mt \end{pmatrix} = \int dq_0 \, \delta(m^2 - q_0^2)\epsilon(q_0) \begin{pmatrix} q_0 \\ m \end{pmatrix} e^{iq_0 t}$$

given by the Dirac measure of the energy-momenta and a frequency q_0 analysis

$$d^4q \, \delta(m^2 - q^2)\epsilon(q_0) = d^3q \, dq_0 \frac{\delta(q_0 - \sqrt{m^2 + \vec{q}^2}) + \delta(q_0 + \sqrt{m^2 + \vec{q}^2})}{2q_0} \tag{121}$$

The measure, integrated with an irreducible representation $e^{iqx} \in \mathbf{U}(1)$ of the additive translation group $x \in \mathbf{R}(2)$, describes the quantization of linear particle fields

$$((0||m^2))(x) = \begin{pmatrix} \mathbf{c}^k(m|x) & i\mathbf{s}(m|s) \\ i\mathbf{s}(m|x) & \mathbf{c}^k(m|x) \end{pmatrix} \tag{122}$$
$$\begin{pmatrix} \mathbf{c}^k(m|x) \\ i\mathbf{s}(m|x) \end{pmatrix} = \int \frac{d^4q}{(2\pi)^3} \delta(m^2 - q^2)\epsilon(q_0) \begin{pmatrix} q^k \\ m \end{pmatrix} e^{iqx}$$

These integrated translation representations are space distributions of representations of the causal group $\mathbf{D}(1)$

$$\mathbf{D}(1) \ni e^{x_0} \mapsto \int d^3x((0||m^2))(x) = (0||m^2)(x_0) = \begin{pmatrix} \delta_0^k \cos mx_0 & i \sin mx_0 \\ i \sin mx_0 & \delta_0^k \cos mx_0 \end{pmatrix} \tag{123}$$

Such a time-oriented analysis and—in the Poincaré group—the Wigner classification are appropriate for the particle interpretation of a field theory (Section 5.3). With respect to the timespace manifold $\mathbf{D}(2) = \mathbf{D}(1) \times \mathbf{SD}(2)$, the representation of the Sylvester factor $\mathbf{SD}(2) \cong \mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$ is not adequately taken into account—this negligence is the main reason for the divergence problems working with linear quantum fields.

The realization of the Cartan group $\mathbf{D}(1) \cong \mathbf{SO}^+(1, 1)$ in the Sylvester factor for linear fields can be seen for the manifolds $\mathbf{D}(1) \times \mathbf{SO}^+(1, s)/\mathbf{SO}(s)$ as follows: The $\mathbf{SO}^+(1, s)$ scalar contribution in the quantization of linear fields (Section 1.2)

$$i\mathbf{s}(m|x) = \int \frac{d^{1+s}q}{(2\pi)^s} \delta(m^2 - q^2)\epsilon(q_0) m e^{iqx} = \frac{\epsilon(x_0)}{\pi i} \int \frac{d^{1+s}q}{(2\pi)^s} m \frac{e^{iqx}}{(q^2 - m^2)_P} \tag{124}$$

shows the time $\mathbf{D}(1)$-representation properties by the space integral

$$\int d^s x \, \mathbf{s}(m|x) = \sin mx_0 \tag{125}$$

whereas the realization of $\mathbf{D}(1)$ in the Sylvester factor for $s \geq 1$ shows up in the ordered time integral

$$\begin{aligned} \int dx_0 \, \epsilon(x_0)\epsilon(m)\mathbf{s}(m|x) &= 2|m| \int \frac{d^s q}{(2\pi)^s} \frac{e^{-i\vec{q}\vec{x}}}{\vec{q}^2 + m^2} \\ &= \begin{cases} e^{-|m\vec{x}|} & \text{for } s = 1 \\ \dfrac{|m|}{2\pi|x|} e^{-|m\vec{x}|} & \text{for } s = 3 \end{cases} \end{aligned} \tag{126}$$

For $s = 1$, the orthogonal manifold $\mathbf{D}(1) \times \mathbf{SO}^+(1, s)/\mathbf{SO}(s)$ is isomorphic to the group $\mathbf{D}(1) \times \mathbf{D}(1)$ with elements $\exp(x_0 - |\vec{x}|)$. Here the exponential $\exp(-|m\vec{x}|)$ represents the space $\mathbf{D}(1) \ni \exp(-|\vec{x}|)$. For $s = 3$, the Yukawa potential is no representation of $\mathbf{D}(1) \subset \mathbf{SD}(2) \cong \mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$.

4.3. Point Measures for Energy-Momenta

For the generalization of the nondecomposable representations $(N|m)$ of the abelian group $\mathbf{D}(1)$ to realizations $(N_1, \ldots, N_n|m_1, \ldots, m_n)$ of the homogeneous timespaces $\mathbf{D}(n)$ with the Cartan subgroups $\mathbf{D}(1)^n$, it is convenient to use the residues of loop integrals for the linear forms $\mathbf{R}(n)^T$ (energy-momenta) of the timespace translations $\mathbf{R}(n)$

$$\frac{(i\lambda\tau)^k}{k!}e^{im\tau} = \int d\mu \frac{\lambda^k}{k!}\delta^{(k)}(m-\mu)e^{i\mu\tau} = \frac{1}{2\pi i}\oint d\mu \frac{\lambda^k}{(\mu-m)^{1+k}}e^{i\mu\tau} \tag{127}$$

with a nontrivial constant $\lambda \in \mathbb{R}$. Therewith one has as integrands for the $(1 + N)$ elements of a $\mathbf{D}(1)$-representation

$$\mathbf{D}(1) \ni e^\tau \mapsto (N|m)(\tau)$$

$$d\mu \frac{\lambda^k}{(\mu-m)^{1+k}}e^{i\mu\tau} \quad \text{with } k = 0, \ldots, N \tag{128}$$

For example, the fundamental complex 2-dimensional $\mathbf{D}(1)$-representations use both poles and dipoles

$$(1|m)(\tau) \cong \begin{pmatrix} 1 & i\lambda\tau \\ 0 & 1 \end{pmatrix} e^{im\tau} = \frac{1}{2\pi i}\oint \frac{d\mu}{\mu-m}\begin{pmatrix} 1 & \dfrac{\lambda}{\mu-m} \\ 0 & 1 \end{pmatrix} e^{i\mu\tau} \tag{129}$$

The structure of the poles, i.e., the location and the order of the singularities, reflects the continuous and the discrete invariant of the $\mathbf{D}(1)$-representation $(N|m)$. The irreducible representations have the *irreducible measures* $d\mu/(\mu - m)$; an additional nontrivial τ dependence is expressed by the *nondecomposable measures* $d\mu\ \lambda^k/(\mu - m)^{1+k}$, reducible for $k = 1, \ldots, N$.

The representation elements of $\mathbf{D}(1)^n$ with Cartan coordinates $\{\xi_r\}_{r=1}^n$ are products of loop integrals

$$\mathbf{D}(1)^n \ni e^\xi = \begin{pmatrix} e^{\xi_1} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & e^{\xi_n} \end{pmatrix} \mapsto (N_1|m_1)(\xi_1) \otimes \cdots \otimes (N_n|m_n)(\xi_n)$$

$$\frac{(i\lambda\xi_1)^{k_1}\cdots(i\lambda\xi_n)^{k_n}}{k_1!\cdots k_n!} e^{i(m_1\xi_1+\cdots+m_n\xi_n)} \tag{130}$$

$$= \frac{1}{(2\pi i)^n}\oint d^n\mu \frac{\lambda^{k_1+\cdots+k_n}e^{i(\mu_1\xi_1+\cdots+\mu_n\xi_n)}}{(\mu_1-m_1)^{1+k_1}\cdots(\mu_n-m_n)^{1+k_n}}$$

$$\text{with } \begin{cases} k_r = 0, \ldots, N_r \\ r = 1, \ldots, n \end{cases}$$

As illustrated in the previous subsection, the conventional relativistic field quantization for $n = 2$ uses only $\mathbf{D}(1)$-representations of the determinant in $\mathbf{U}(1)$

$$\mathbf{D}(1) \ni \det e^{\xi} = e^{\xi_1 + \cdots + \xi_n} = e^{x_0} \mapsto e^{imx_0} \in \mathbf{U}(1) \tag{131}$$

The correspondingly frugal analogue for the compact group $\mathbf{U}(n)$ is given by its $\mathbf{U}(1)$-representations with only one winding number $Z \in \mathbb{Z}$

$$\mathbf{U}(n) \supseteq \mathbf{U}(1)^n \ni e^{i\beta} = \begin{pmatrix} e^{i\beta_1} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & e^{i\beta_n} \end{pmatrix} \mapsto e^{iZ(\beta_1 + \cdots + \beta_n)} \in \mathbf{U}(1) \tag{132}$$

unfaithful for $n \geq 2$.

If the Cartan subgroup $\mathbf{D}(1)^n$ is realized in the homogeneous timespace $\mathbf{D}(n)$

$$\mathbf{D}(1) \to \mathbf{D}(1)^n \hookrightarrow \mathbf{D}(n) \tag{133}$$

using Lebesque measures for the tangent structures

$$d\mu \text{ on } \mathbb{R} \to d^n\mu = d\mu_1 \cdots d\mu_n \text{ on } \mathbb{R}^n \hookrightarrow d^{n^2}q \text{ on } \mathbb{R}(n)^T \tag{134}$$

the n Cartan masses $(m_1, \ldots, m_n)$ for the $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$-weights come as poles of the $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-invariant determinant $\det q = q^n$ with the nth powers m^n of the causal masses

$$\frac{d\mu}{\mu - m} e^{i\mu\tau} \to \frac{d^n\mu}{(\mu_1 - m_1)\cdots(\mu_n - m_n)} e^{i\mu_i\xi_i} \hookrightarrow \frac{d^{n^2}q}{(q^n - m^n_1)\cdots(q^n - m^n_n)} e^{iqx} \tag{135}$$

In the conventional field quantization for $n = 2$ only one continuous invariant is used (Section 1.2)

$$\frac{d\mu}{\mu - m} e^{i\mu x_0} \hookrightarrow \frac{d^{n^2}q}{q^n - m^n} e^{iqx} \tag{136}$$

The invariance group of the *irreducible point measures* for the $\mathbf{D}(n)$-realizations

$$\frac{d^{n^2}q}{(q^n - m^n)\cdots(q^n - m^n_n)}, \quad (m_1, \ldots, m_n) \in \mathbb{R}^n \tag{137}$$

is the adjoint group $\mathbf{SL}(\mathbb{C}^n)/\mathbf{I}_n$. The singularities arise for the invariant energy momenta

$$\mu(q) = \epsilon(q)|(\det q)^{\frac{1}{n}}| = \begin{cases} \mu & \text{for } n = 1 \\ \epsilon(q_0)\vartheta(q^2)\sqrt{q^2} & \text{for } n = 2 \end{cases} \tag{138}$$

i.e., one real pole for odd rank n of timespace and two real poles with opposite sign for even rank

$$\begin{aligned} &\mu^n - m^n = (\mu - m)(\mu^{n-1} + m\mu^{n-2} + \cdots + m^{n-1}) \\ &\{\mu(q) \in \mathbb{R} | \mu(q)^n = m^n = \begin{cases} \{m\} & \text{for } n = 1, 3, \ldots \\ \{\pm m\} & \text{for } n = 2, 4, \ldots \end{cases} \subseteq m\mathbb{I}_n \end{aligned} \tag{139}$$

Therewith for even rank, e.g., for the relativistic case, the representations are m^2-dependent.

The nonabelian compact properties in $\mathbf{D}(n) \cong \mathbf{D}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}$, nontrivial for $n \geq 2$, have to be realized via tensor product polynomials $q \otimes \ldots \otimes q$ of degree $2J \in \mathbb{N}$ in the energy-momenta, placed in the numerator of the integrand

$$\frac{\lambda^k}{(\mu - m)^k} \to \frac{\lambda^{k_1 + \cdots + k_n}}{(\mu_1 - m_1)^{1+k_1} \cdots (\mu_n - m_n)^{1+k_n}} \hookrightarrow \frac{\lambda^{k_1+\cdots+k_n}(q \otimes \cdots \otimes q)_{2J \text{ times}}}{(q^n - m_1^n)^{1+k_1} \cdots (q^n - m_n^n)^{1+k_n}}$$
$$\text{with} \quad 2J = (n-1)(k_1 + \cdots + k_n) \tag{140}$$

Therewith the relevant integrands for the realization of a timespace point $d \in \mathbf{D}(n)$ are given with its translations $x = \log d \in \mathbb{R}(n)$

$$\begin{aligned} &\mathbf{D}(n) \ni e^x \mapsto (N_1, \ldots, N_n | m_1, \ldots, m_n)(x) \\ &d^{n^2}q \frac{\lambda^{k_1+\cdots+k_n}(q \otimes \cdots \otimes q)_{2J \text{ times}}}{(q^n - m_1^n)^{1+k_1} \cdots (q^n - m_n^n)^{1+k_n}} e^{iqx} \\ &\quad \text{with} \begin{cases} k_r = 0, \ldots, N_r \\ r = 1, \ldots, n \\ 2J = (n-1)(k_1 + \cdots + k_n) \end{cases} \end{aligned} \tag{141}$$

4.4. Realizations of Relativistic Timespace

The representations of the rank 2 relativistic timespace manifold $\mathbf{D}(2) \cong \mathbf{D}(1)^2 \times \mathbf{SU}(2)/\mathbf{U}(1)$ involve two Cartan masses $(m_1, m_2) \in \mathbb{R}^2$ for the Cartan subgroup $\mathbf{D}(1)^2$,

$$\mathbf{D}(2) \ni e^x \mapsto (N_1, N_2|m_1, m_2)(x)$$

$$d^4q \, \frac{\lambda^{k_1+k_2}(q \otimes \cdots \otimes q)_{(k_1+k_2)\text{ times}}}{(q^2 - m^2)^{1+k_1}(q^2 - m_2^2)^{1+k_2}} \, e^{iqx} \quad \text{with} \quad \begin{cases} k_1 = 0, \ldots, N_1 \\ k_2 = 0, \ldots, N_2 \end{cases} \tag{142}$$

The scalar representations use the irreducible scalar measures

$$\mathbf{D}(2) \ni e^x \mapsto (0, 0|m_1, m_2)(x)$$

$$\frac{d^4q}{(q^2 - m_1^2)(q^2 - m_2^2)} \, e^{iqx} \tag{143}$$

leading to the well-behaved explicit functions (Section 1.2)

$$\begin{aligned} (0, 0|m_1, m_2)(x) &= \frac{\epsilon(x_0)}{\pi} \int \frac{d^4q}{\pi^2} \, \frac{1}{(q^2 - m_1^2)_P (q^2 - m_2^2)_P} \, e^{iqx} \\ &= \epsilon(x_0)\vartheta(x^2) \, \frac{m_1^2 \mathscr{E}_1(m_1^2 x^2) - m_2^2 \mathscr{E}_1(m_2^2 x^2)}{m_1^2 - m_2^2} \\ (0, 0|m, m)(x) &= \epsilon(x_0)\vartheta(x^2)\mathscr{E}_0(m^2 x^2) \\ (0, 0|0, 0)(x) &= \epsilon(x_0)\vartheta(x^2) \end{aligned} \tag{144}$$

The principal value pole integration (denoted by P) has been used. The massless case with the trivial realization is particularly simple.

The reducible, but nondecomposable realizations with $(N_1, N_2) = (0, 1)$ are faithful and nontrivial for both the abelian group $\mathbf{D}(1)$ and the nonabelian boost manifold $\mathbf{SL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{SU}(2)$—they involve both poles and dipoles

$$\mathbf{D}(2) \ni e^x \mapsto (0, 1|m_1, m_2)\,(x)$$

$$\frac{d^4q}{(q^2 - m_1^2)(q^2 - m_2^2)} \begin{pmatrix} 1 & \dfrac{\lambda q}{q^2 - m_2^2} \\ 0 & 1 \end{pmatrix} e^{iqx} \tag{145}$$

Here one has the explicit functions with the reducible measure

$$\begin{aligned} &\frac{\epsilon(x_0)}{\pi} \int \frac{d^4q}{\pi^2} \, \frac{\lambda q}{(q^2 - m_1^2)_P (q^2 - m_2^2)_P^2} \, e^{iqx} \\ &= \epsilon(x_0)\vartheta(x^2) \, \frac{ix\lambda}{4} \left[\frac{m_1^4 \mathscr{E}_2(m_1^2 x^2) - m_2^4 \mathscr{E}_2(m_2^2 x^2)}{(m_1^2 - m_2^2)^2} - 2 \, \frac{m_2^2 \mathscr{E}_1(m_2^2 x^2)}{m_1^2 - m_2^2} \right] \end{aligned} \tag{146}$$

and the special cases for coinciding and trivial masses

$$\begin{aligned}(0, 1|m, m)(x) &= \epsilon(x_0)\vartheta(x^2)\begin{pmatrix}1 & i\lambda x/4\\ 0 & 1\end{pmatrix}\mathscr{E}_0(m^2x^2)\\ (0, 1|0, 0)(x) &= \epsilon(x_0)\vartheta(x^2)\begin{pmatrix}1 & i\lambda x/4\\ 0 & 1\end{pmatrix}\end{aligned} \qquad (147)$$

5. FUNDAMENTAL QUANTUM FIELDS

In quantum structures, representations of causal timespace manifolds $\mathbf{D}(n)$ are parametrized by operators, i.e., quantum variables: time-dependent positions and momenta in quantum mechanics, timespace-dependent fields in relativistic quantum field theory.

Nondecomposable representations $(N|m)$ of the causal group $\mathbf{D}(1)$ are parametrizable by principal vectors, the irreducible ones $(0|m)$ in $\mathbf{U}(1)$ even by eigenvectors (Section 1.1) There exist pairs of *cyclic principal vectors* $(\mathrm{b}, \mathrm{b}^\times)$ in the representation space and its dual V, $V \cong \mathbb{C}^{1+N}$, which are $\mathbf{U}(N_+, N_-)$-conjugated to each other and parametrize the characteristic matrix element[17] of the representation

$$\langle \mathrm{b}^\times, \mathrm{b}\rangle(\tau) = \frac{(i\lambda\tau)^N}{N!}e^{im\tau} \text{ e.g., } N = 0:\quad \langle \mathrm{b}^\times, \mathrm{b}\rangle(\tau) = \langle \mathrm{u}^*, \mathrm{u}\rangle(\tau) = e^{im\tau}$$
$$N = 1:\quad \langle \mathrm{b}^\times, \mathrm{b}\rangle(\tau) = i\lambda\tau e^{im\tau} \qquad (148)$$

Only for the irreducible representations are the cyclic vectors unique; in general, they are 'gauge dependent' (Saller, 1991, 1993b).

Generalized for timespaces $\mathbf{D}(n)$ with $n \geq 1$, such pairs of cyclic principal vectors—used as operators (Saller, 1993a)—will be called *fundamental time-space quantum operators.*

5.1. The Fundamental Mechanical Pair

Quantum mechanics, $n = 1$, uses one kind[18] of fundamental pair—a creation–annihilation pair $(\mathrm{u}, \mathrm{u}^*)$ in the complex or a position–momentum pair (X, iP) in the 'real' formulation—for the causal group $\mathbf{D}(1)$-representations in $\mathbf{U}(1)$ and $\mathbf{SO}(2)$, respectively:

[17] With $v \in V$ and $\omega \in V^T$ the short-hand notation $\langle\omega(\tau_2), v(\tau_1)\rangle = \langle\omega, v\rangle(\tau_1 - \tau_2)$ for the time-dependent dual product is used.

[18] The basically unimportant number of fundamental pairs $[iP_a, X_b] = \delta_{ab}$, $a, b = 1, \ldots, N$ e.g., $N = 3$ for the 3-dimensional isotropic oscillator or the quantum mechanical nonrelativistic hydrogen atom give rise to decomposable $\mathbf{D}(1)$ representations with 'internal' degrees of freedom, e.g., with respect to $\mathbf{SU}(3)$ (oscillator) and $\mathbf{SO}(4)$ (bound states for the atoms), respectively.

$$\mathbf{D}(1) \ni e^t \mapsto \begin{cases} [\mathrm{u}^*, \mathrm{u}](t) = e^{imt} = (0|m)(t) \\ \begin{pmatrix} [iP, X] & [X, X] \\ [P, P] & [X, -iP] \end{pmatrix}(t) = \begin{pmatrix} \cos mt & \frac{i}{Mm} \sin mt \\ iM\, m \sin mt & \cos mt \end{pmatrix} \cong (0||m^2)(t) \end{cases} \tag{149}$$

The $\mathbf{D}(1)$ representation is generated by the Hamiltonian $H = P^2/(2M) + \kappa X^2/2$.

Representations of $\mathbf{D}(1)$ with a normalized positive frequency measure

$$\mathbf{D}(1) \ni e^t \mapsto \begin{cases} \langle[\mathrm{u}^*, \mathrm{u}]\rangle(t) = \int d\mu\, h_0(\mu) e^{i\mu t} \\ \begin{pmatrix} \langle[iP, X]\rangle & \langle[X, X]\rangle \\ \langle[P, P]\rangle & \langle[X, -iP]\rangle \end{pmatrix}(t) = \int d\mu^2\, h_0(\mu^2) \begin{pmatrix} \cos \mu t & \frac{1}{M\mu} \sin \mu t \\ iM\mu \sin \mu t & \cos \mu t \end{pmatrix} \end{cases}$$

$$\text{for } t = 0: \begin{cases} \langle[\mathrm{u}^*, \mathrm{u}]\rangle(0) = [\mathrm{u}^*, \mathrm{u}] = \int d\mu\, h_0(\mu) = 1 \\ \langle[iP, X]\rangle(0) = [iP, X] = \int d\mu^2\, h_0(\mu^2) = 1 \end{cases} \tag{150}$$

arise for the time developments of the ground-state values of the commutators in the case of Hamiltonians $H = P^2/(2M) + V(X)$ which lead to bound states only.

There may also occur indefinite unitary nondecomposable representations of the causal group, if there arise not only bound states, e.g., for a free mass point with Hamiltonian $H = P^2/(2M)$

$$\mathbf{D}(1) \ni e^t \mapsto \begin{pmatrix} [iP, X] & [X, X] \\ [P, P] & [X, -iP] \end{pmatrix}(t) \cong \begin{pmatrix} 1 & it/M \\ 0 & 1 \end{pmatrix} \cong (1|0)(t) \in \mathbf{U}(1, 1) \tag{151}$$

5.2. The Fundamental Quantum Fields

Many linear fields are used for the Minkowski translations $\mathbb{R}(2)$, e.g., lepton, quark, and gauge fields in the standard model of elementary particles. Those fields, appropriate for a free theory, do not parametrize realizations $(N_1, N_2|m_1, m_2)$ of the causal timespace manifold $\mathbf{D}(2)$, but space distributions of time $\mathbf{D}(1)$ representations (Section 4.2) with possible spin degrees of freedom (Saller, n.d.).

For example, a massive Dirac field (Section 1.2) parametrizes the space distribution of the $\mathbf{U}(1)$ time representation

$$\begin{aligned} &\mathbf{D}(1) \ni e^{x_0} \mapsto \{\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}\}(x) = \mathbf{exp}(im|x) = \gamma_k \mathbf{c}^k(m|x) + i\mathbf{s}(m|x) \\ &\text{for } \; x_0 = 0: \quad \{\overline{\boldsymbol{\Psi}}, \boldsymbol{\Psi}\}(\vec{x}) = \gamma_0 \delta(\vec{x}) \end{aligned} \tag{152}$$

In analogy to the fundamental bosonic pairs $(u, u^\star)$ (quantum mechanics) with one causal unit m, *fundamental fermionic pairs*[19] $(b^A, b^\times_A)_{A=1,2}$ with fundamental conjugated $\mathbf{SL}(\mathbb{C}^2_\mathbb{R})$-representations $[1|0]$ and $[0|1]$ are proposed (Heisenberg, 1967) for the timespace manifold $\mathbf{D}(2)$ (quantum field theory) with two causal units $m_{1,2}$. They are assumed to have as dual product (quantization) a faithful $\mathbf{D}(2) \cong \mathbf{D}(1) \times \mathbf{SL}(\mathbb{C}^2_\mathbb{R})/\mathbf{SU}(2)$ realization $(0, 1|m_1, m_2)$

$$\mathbf{D}(2) \ni e^x \mapsto (0, 1|m_1, m_2)(x)$$
$$\text{with } \{b^\times, b\}(x) = \frac{\epsilon(x_0)}{\pi i} \int \frac{d^4q}{\pi^2} \frac{q}{(q^2 - m_1^2)_P (q^2 - m_2^2)_P^2} e^{iqx} \tag{153}$$

The characteristic property of the fundamental fields is the representation of the external and internal operations, used for the definition of the timespace manifold $\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}^2_\mathbb{R})/\mathbf{U}(2)$, *not some linear field equation.* The tangent flat timespace interpretation with linear particle fields, i.e., with positive unitary representations of the Poincaré group (Wigner, 1939) $\mathbf{SO}^+(1, 3) \times_s \mathbb{R}(2)$ for a given dynamics and the determination of a measure $h(\mu_1^2, \mu_2^2)$ for a decomposable realization—in analogy to $h(\mu^2)$ in the previous subsection—is the essential part of the solution of a dynamics.

A parametrization of the causal measure with point supported realizations may be useful for a perturbative approach to generate the causal subalgebra associated with a given dynamics. Being part of the causal measure algebra, the product representations are well defined, i.e., there arise no divergences.

5.3. Particles in Fundamental Fields

As familiar from regularization recipes, the price to be paid for the relativistic causal representations with convolution properties (no divergences) is the indefinite metric for some representations of the timespace translations $\mathbb{R}(2)$, i.e., an indefinite conjugation, e.g., $\mathbf{U}(1, 1)$ (Section 3.1). This price would be too high if it invalidated a probability interpretation of the experimental consequences. A closer analysis of the manifold $\mathbf{D}(2)$ and its tangent Minkowski translations $\mathbb{R}(2)$ reveals a rather subtle situation: The causal und the spacelike submanifolds $\mathbb{R}(2)_{\text{causal}}$ and $\mathbb{R}(2)_{\text{space}}$, respectively (Section 2.1) are reflected in the two parts of Feynman propagators (Section 1.2). The field quantization realizes the timespace manifold $\mathbf{D}(2)$ and is supported by the causal submanifold of flat timespace $\mathbb{R}(2)$. The asymptotic consequences of a dynamics, however, are interpreted with the spacelike submanifold for in- and outgoing particles. The spacelike submanifold arises

[19] For a full-fledged parametrization, also $\mathbf{U}(2)$ internal hyperisospin degrees of freedom have to be introduced to take care of possibly nontrivial properties of the coset $\mathbf{U}(2)$ in the timespace manifold $\mathbf{GL}(\mathbb{C}^2_\mathbb{R})/\mathbf{U}(2)$.

only as part of the tangent translations $\mathbb{R}(2)$ on the timespace manifold $\mathbf{D}(2)$ (Section 2.1).

The fundamental quantum fields introduced in Section 5.2 cannot be expanded in terms of energy-momenta eigenvectors only, i.e., with positive $\mathbf{U}(1)$ representations of the timespace translations $\mathbb{R}(2)$. The causal spreading on four dimensions leads to negative and derived Dirac measures of the energy-momenta $\mathbb{R}(2)^T$, e.g., $\delta'(m_2^2 - q^2)$, related to reducible, but nondecomposable representations used for the 'hyperbolic stretching' group $\mathbf{SO}^+(1, 1) \cong \mathbf{D}(1)$. Therefore, the fundamental fields contain, on the one hand, positive-definite $\mathbf{U}(1)$-representations of the timespace translations, describing particles, e.g., leptons; on the other hand they involve indefinite, e.g., $\mathbf{U}(1, 1)$-representations of the translations which describe interactions via fields without particle content. Examples for nonparticle fields are the Coulomb interactions (no energy eigenvectors) in the four-component electromagnetic field (Nakanishi and Ojima, 1990) $\mathbf{A}^k(x)$ which has only two particle degrees of freedom (the photons with left- and right-handed polarization). The quantization of the electromagnetic field involves $\mathbf{U}(1)$ and $\mathbf{U}(1, 1)$ representations of the translations (Saller, *et al.*, 1995) as reflected by the underived and derived point measures

$$[\mathbf{A}^k, \mathbf{A}^j](x) = \int \frac{d^4q}{(2\pi)^3} [-e_0^2\eta^{jk}\delta(q^2) - \lambda_0 q^k q^j \delta'(q^2)]\epsilon(q_0)e^{iqx} \quad (154)$$

with $e_0^2/4\pi$ the fine structure constant and λ_0 a gauge-fixing parameter. The Fadeev–Popov ghosts are another example for indefinite unitary representations of the translations (Saller, n.d.).

The spacelike (asymptotic) behavior of a particle field $\mathbf{\Psi}$ (Section 1.2) is completely given by the Fock value of the 'quantization-opposite' commutator, e.g., for the electron field

$$\text{for } x^2 < 0: \quad \langle \mathscr{C}\overline{\mathbf{\Psi}}\mathbf{\Psi}\rangle(x) = \langle[\overline{\mathbf{\Psi}}, \mathbf{\Psi}]\rangle(x)$$
$$= \mathbf{EXP}(im|x) = \mathbf{C}(m|x) + i\gamma_k\mathbf{S}^k(m|x) \quad (155)$$

The spacelike behavior is decisive for the probability interpretation—here the indefinite metric has to be avoided. In- and outgoing particles are tangent spacelike phenomena.

The spacelike behavior of particles is described explicitly by

$$\begin{pmatrix} \mathbf{C}(m|x) \\ i\mathbf{S}^k(m|x) \end{pmatrix} = \frac{m^2}{4\pi^2}\epsilon(m)m\begin{pmatrix} -1 \\ 4\dfrac{\partial}{\partial m x_k} \end{pmatrix}$$
$$\times \frac{\partial}{\partial m^2x^2}\left[\log\frac{|m^2x^2|}{4}\mathscr{E}_0(m^2x^2) - \mathscr{F}_0(m^2x^2)\right] \quad (156)$$

In addition to the functions $\mathscr{E}_n$, given in Section 1.2, it involves the functions

$$\begin{aligned} \mathscr{F}_n(\xi^2) &= n! \sum_{j=0}^{\infty} \frac{(-\xi^2/4)^j}{j!(j+n)!} [\varphi(j+n) - 2\gamma_0)], \\ j \geq 1: \quad \varphi(j) &= 1 + \frac{1}{2} + \cdots \frac{1}{j} \\ \gamma_0 = \frac{\varphi(0)}{2} &= \lim_{j\to\infty} [\varphi(j) - \log j] = 0.5772\cdots \quad \text{(Euler's constant)} \end{aligned} \tag{157}$$

For relativistic timespace $\mathbf{D}(2)$, the quantization distributions $\mathbf{c}^k(m|x)$, $\mathbf{s}(m|x)$, and $\mathbf{exp}(im|x)$, on the one hand, and the Fock functions $\mathbf{C}(m|x)$, $\mathbf{S}^k(m|x)$, and $\mathbf{EXP}(im|x)$, on the other, are completely different, whereas their analogues coincide for time $\mathbf{D}(1)$ (Section 1.1)

$$\begin{aligned} \begin{pmatrix} \cos mt \\ i \sin mt \end{pmatrix} &= \int dq_0\, \delta(m^2 - q_0^2)\epsilon(q_0)\begin{pmatrix} q_0 \\ m \end{pmatrix} e^{iq_0 t} \\ &= \int dq_0\, \delta(m^2 - q_0^2)\epsilon(m)\begin{pmatrix} m \\ q_0 \end{pmatrix} e^{iq_0 t} \end{aligned} \tag{158}$$

The Fock functions of the position-momentum pair for irreducible causal representations in $\mathbf{U}(1)$ are exemplified by the harmonic oscillator with intrinsic length $l^4 = 1/\kappa M$,

$$\langle\{\mathrm{u}^\star, \mathrm{u}\}\rangle(t) = e^{imt} \tag{159}$$

$$\begin{pmatrix} \langle\{iP, X\}\rangle & \langle\{X, X\}\rangle \\ \langle\{P, P\}\rangle & \langle\{X, -iP\}\rangle \end{pmatrix}(t) = \begin{pmatrix} 0 & l^2 \\ 1/l^2 & 0 \end{pmatrix}\begin{pmatrix} \cos mt & il^2 \sin mt \\ \frac{1}{l^2}\sin mt & \cos mt \end{pmatrix}$$

The value for $t = 0$ reflects the $\mathbf{U}(1)$-scalar product for the creation–annihilation operator pair $(\mathrm{u}, \mathrm{u}^\star)$

$$\langle \mathrm{u}^\star \mathrm{u}\rangle = 1, \qquad \langle \mathrm{u}\mathrm{u}^\star\rangle = 0 \Rightarrow 2\langle X^2\rangle = l^2, \qquad 2\langle P^2\rangle = \frac{1}{l^2} \tag{160}$$

Going from time translations $\mathbb{R}(1)$ to relativistic timespace translations $\mathbb{R}(2)$, the point $\mathbb{R}(1)_{\text{space}} = \{0\}$ (presence) is 'blown up' into the spacelike submanifold $\mathbb{R}(2)_{\text{space}}$. For particle fields, i.e., with $\mathbf{U}(1)$-representations of the translations, e.g., for the chiral components of a Dirac electron field $\boldsymbol{\Psi}(x) = \binom{\mathbf{l}(x)}{\mathbf{r}(x)}$, the creation and annihilation operators for the particle $(u(\vec{q}), u^\star(\vec{q}))$ and antiparticle $(a(\vec{q}), a^\star(\vec{q}))$ involved

$$\mathbf{l}(x) = \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{|m|}{|q_0|}}\, e^{\vec{\beta}\vec{\sigma}/2}\, \frac{e^{ixq}\mathrm{u}(\vec{q}) + e^{-ixq}\mathrm{a}^\star(\vec{q})}{\sqrt{2}}$$

$$\mathbf{r}(x) = \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{|m|}{|q_0|}}\, e^{-\vec{\beta}\vec{\sigma}/2}\, \frac{e^{ixq}\mathrm{u}(\vec{q}) - e^{-ixq}\mathrm{a}^\star(\vec{q})}{\sqrt{2}} \tag{161}$$

$$\text{with} \quad q = (|q_0|, \vec{q}), \qquad |q_0| = \sqrt{m^2 + \vec{q}^{\,2}}$$

$$\vec{\beta} = \frac{\vec{q}}{|\vec{q}|} \operatorname{Artanh} \frac{|\vec{q}|}{|q_0|}$$

have the $\mathbf{U}(1)$-scalar product

$$\begin{aligned} \langle \mathrm{u}^\star(\vec{p})\mathrm{u}(\vec{q})\rangle &= \langle \mathrm{a}^\star(\vec{p})\mathrm{a}(\vec{q})\rangle = (2\pi)^3\delta(\vec{q} - \vec{p}) \\ \langle \mathrm{u}(\vec{p})\mathrm{u}^\star(\vec{q})\rangle &= \langle \mathrm{a}(\vec{p})\mathrm{a}^\star(\vec{q})\rangle = 0 \end{aligned} \tag{162}$$

A rest system, used for the experiments, defines a decomposition of $\mathbb{R}(2)$ into time and space translations and a Lebesque measure factorization $d^4q = d^3q dq_0$. A corresponding representation of the spacelike behavior shows the spherical waves for the particles

$$\mathbf{C}(m|\vec{x}) = \frac{m^2}{(2\pi)^2} \int dq_0\, \vartheta(q_0^2 - m^2)\, \frac{\sin\sqrt{q_0^2 - m^2}|\vec{x}|}{|m\vec{x}|}, \qquad d^3x\, \mathbf{C}(m|\vec{x}) = 1 \tag{163}$$

Only the $\mathbf{U}(1)$-representations of the translations have a positive-definite scalar product as expressed by the Fock state. The indefinite unitary representations have an indefinite inner product (Section 3.2), as characterized by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $\mathbf{U}(1, 1)$. Here the abelian $\mathbf{U}(1)$. Fock form is inappropriate. It has to be replaced by the nonabelian Heisenberg–Killing form (Saller, 1992a), leading to trivial spacelike contributions in the Feynman propagator.

In the nondecomposable point measure approximation for the fundamental fields for the realization of the timespace manifold only the $\mathbf{U}(1)$-representation of the translations can give rise to in- and outgoing particles. This is effected by the familiar Feynman integration prescription for a particle at its pole in the complex energy–momentum plane using a limit vanishing imaginary part $m_1^2 - io$ (Section 1.2)

$$\langle \mathscr{C}\mathrm{b}^\times \mathrm{b}\rangle(x) = \frac{i}{\pi} \int \frac{d^4q}{\pi^2}\, \frac{q}{(q^2 - m_1^2 + io)(q^2 - m_2^2)_P^2}\, e^{iqx} \tag{164}$$

The representations of the translations which cannot be spanned by energy–momentum eigenvectors, i.e., related to the reducible, but nondecomposable representations—here the dipole—are confined to the causal submanifold with the principal value integration. They give rise to interactions, not, however, to particles.

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